# 9 Incomplete Block Designs: An Introduction

It is sometimes necessary to block experimental units into groups smaller than a complete replication of all treatments with a randomized complete block or Latin square design as illustrated in Chapter 8. The incomplete block design is utilized to decrease experimental error variance and provide more precise comparisons among treatments than is possible with a complete block design. A general description of some major groups of incomplete block designs is presented in this chapter. The method of randomization and basic analysis methods are demonstrated for balanced and partially balanced incomplete block designs. The efficiency of the designs is also considered.

## 9.1 Incomplete Blocks of Treatments to Reduce Block Size

Experiments can require a reduction in block size for one of several reasons. Complete block designs can reduce the estimate of experimental error variances, but sometimes the reduction is insufficient. Alternatively, the number of treatments may be so large as to render a complete block design impractical for reducing experimental error variance. Also, the natural grouping of experimental units into blocks can result in fewer units per block than required by the number of treatments for a complete block design. In the following example, limited numbers of environmental control chambers prevented the performance of a complete replication of all treatments in one run of the available chambers.





**Example 9.1 Tomato Seed Germination at Constant High Temperature**

Tomatoes often are produced during the winter months in arid and tropical regions. Winter production requires seeding during late summer when soil temperatures can exceed 40°C, which surpasses the suggested maximum germination temperature of 35°C for tomato seed.

***Research Objective*:** A plant scientist wanted to determine in what temperature range she could expect inhibition of tomato seed germination for a group of tomato cultivars.

***Treatment Design*:** Four temperatures were chosen to represent a temperature range common for the cultivation area under consideration. They were 25°C, 30°C, 35°C, and 40°C. The tomato seed was to be subjected to a constant temperature in a controlled environment chamber.

***Experiment Design*:** A single chamber would be an experimental unit since true replication of any temperature treatment required a separate run of the temperature treatment in a chamber. Any number of factors could contribute to variation in response between runs since the entire experimental setup had to be repeated for a replicate run. Thus, blocking on runs was considered essential.

One complete block and replication of the experiment required four chambers; however, only three chambers were at the disposal of the plant scientist. Since the natural block of one run had fewer chambers (experimental units) than treatments, she constructed an incomplete block design.

A diagram of the design is shown in Display 9.1. Three different temperatures were tested in each of the four runs. The runs represent incomplete blocks of three temperature treatments. The treatments were randomly assigned to the chambers in each run. Some special features of this design are discussed in the next section.

**Display 9.1 An Incomplete Block Design with Four Treatments in Blocks of Three Units**

| | *Chamber* | | | | *Chamber* | | |
|---|---|---|---|---|---|---|---|
| | *1* | *2* | *3* | | *1* | *2* | *3* |
| *Run 1* | 25° | 30° | 40° | *Run 2* | 35° | 30° | 25° |
| | *1* | *2* | *3* | | *1* | *2* | *3* |
| *Run 3* | 40° | 25° | 35° | *Run 4* | 40° | 30° | 35° |

*Source*: Dr. J. Coons, Department of Botany, Eastern Illinois University.

*Other Examples*: Batches of material for industrial research serve as blocks, but insufficient material may exist in a single batch for all experimental treatments. The criteria for matching subjects may result in insufficient numbers of like subjects or cohorts in each group to accommodate the treatments planned for the study. Agronomic variety trials often contain large numbers of varieties for test and complete blocks are not practical for reduction of error variance. Incomplete block designs are suitable choices for the experiments in each of these examples.

The guiding principle for block size is to have a homogeneous set of experimental units for more precise treatment comparisons. Incomplete block designs were introduced by Yates (1936a, 1936b) for experiments in which the number of experimental units per block are less than the number of treatments. The designs were developed from a need for experiments that included the relevant set of treatments to address the research hypotheses yet were constrained to sensible block sizes.

The incomplete block designs may be classified into two major groups of designs: those arranged in randomized incomplete blocks with one blocking criterion and those with arrangements based on Latin squares with two blocking criteria. The designs may be balanced wherein each treatment is paired an equal number of times with every other treatment in the same blocks somewhere in the experiment. A partially balanced design occurs when different treatment pairs occur in the same blocks an unequal number of times or some treatment pairs never occur together in the same block. An overview of incomplete block designs will be presented in this chapter along with an introduction to the analysis of data from these designs.

## 9.2 Balanced Incomplete Block (BIB) Designs

### The BIB Design Compares All Treatments with Equal Precision

The **balanced incomplete block design** is arranged such that all treatments are equally replicated and each treatment pair occurs in the same block an equal number of times somewhere in the design. The balance obtained from equal occurrence of all treatment pairs in the same block results in equal precision for all comparisons between pairs of treatment means.

The incomplete block design has $r$ replications of $t$ treatments in $b$ blocks of $k$ experimental units with $k < t$. The total number of experimental units is $N = rt = bk$. The design for the tomato experiment in Display 9.1 has $b = 4$ blocks of $k = 3$ experimental units. Each of the $t = 4$ treatments is replicated $r = 3$ times. There are a total of $N = bk = 4 \cdot 3 = 12$ or $N = rt = 3 \cdot 4 = 12$ experimental units. Upon inspection it can be seen that each treatment pair occurs together in the same block twice. For example, the treatment pair (25°, 30°) occurs in blocks 1 and 2 and the treatment pair (30°, 35°) occurs in blocks 2 and 4.

The number of blocks in which each pair of treatments occurs together is $\lambda = r(k-1)/(t-1)$, where $\lambda < r < b$. The integer value $\lambda$ derives from the fact that each treatment is paired with the other $t - 1$ treatments somewhere in the design $\lambda$ times. There are $\lambda(t-1)$ pairs for a particular treatment in the experiment. The same treatment appears in $r$ blocks with $k - 1$ other treatments, and each treatment appears in $r(k-1)$ pairs. Therefore,

$$\lambda(t-1) = r(k-1) \quad \text{or} \quad \lambda = r(k-1)/(t-1)$$

For the tomato experiment design in Display 9.1, $\lambda = 3(3-1)/(4-1) = 2$.

A balanced incomplete block design can be constructed by assigning the appropriate combinations of treatments to each of $b = \binom{t}{k}$ blocks to achieve a balanced design. Frequently, balance is possible with less than $\binom{t}{k}$ blocks.

There is no single method for constructing all classes of balanced incomplete block designs. Methods do exist for constructing certain classes of incomplete block designs. The topic of design construction has been the subject of much mathematical research, resulting in a vast array of balanced and partially balanced incomplete block designs.

Plans for small numbers of treatments are given in Appendix 9A.1. Additional tables of designs useful for many practical situations can be found in Cochran and Cox (1957) and Fisher and Yates (1963). Several important categories of traditional balanced incomplete block designs will be illustrated in more detail in Chapter 10. An introduction to the basic structure of incomplete block designs with illustrations of their application and analysis is the focus of this chapter.



## 9.3 How to Randomize Incomplete Block Designs

After the basic design has been constructed with the treatment code numbers, the steps in randomization follow:

*Step 1.* Randomize the arrangement of the blocks of treatment code number groups.

*Step 2.* Randomize the arrangement of the treatment code numbers within each block.

*Step 3.* Randomize the assignment of treatments to the treatment code numbers in the plan.

Randomization is illustrated with the basic design plan for $t = 4$ treatments in $b = 4$ blocks of $k = 3$ experimental units each. Prior to randomization the plan is

| *Block* | | | |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 1 | 2 | 4 |
| 3 | 1 | 3 | 4 |
| 4 | 2 | 3 | 4 |

*Step 1.* Suppose the blocks for the experiment are the runs of three growth chambers with three temperature treatments used in Example 9.1. The treatment groups (1, 2, 3), (1, 2, 4), (1, 3, 4), and (2, 3, 4) must be randomly assigned to the runs. Choose a random permutation of the numbers 1 to 4, and assign the four blocks to the four runs. With the random permutation 2, 4, 1, 3 the assignment is

| *Run* | | | | *Original Block* |
|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 2 |
| 2 | 2 | 3 | 4 | 4 |
| 3 | 1 | 2 | 3 | 1 |
| 4 | 1 | 3 | 4 | 3 |

*Step 2.* Assign random treatment code numbers to the three growth chambers in each run. Choose a random permutation of the numbers 1 to 4 for each chamber, and omit the treatment number absent in the run. Four random permutations along with the assignment to each chamber (A, B, C) in each run follow:

| | *Chamber* | | | *Permutation* | | | |
|---|---|---|---|---|---|---|---|
| *Run* | A | B | C | | | | |
| 1 | 2 | 4 | 1 | 2 | 4 | ~~3~~ | 1 |
| 2 | 3 | 4 | 2 | 3 | 4 | ~~1~~ | 2 |
| 3 | 1 | 2 | 3 | ~~4~~ | 1 | 2 | 3 |
| 4 | 1 | 4 | 3 | 1 | 4 | 3 | ~~2~~ |

*Step 3.* Suppose the treatments are temperatures of 25°C, 30°C, 35°C, and 40°C. A random permutation 2, 4, 3, 1 gives a random assignment of temperatures to treatment code numbers with the replacements 2→25°C, 4→30°C, 3→35°C, and 1→40°C in the previous display.

| | *Chamber* | | |
|---|---|---|---|
| *Run* | A | B | C |
| 1 | 25°C | 30°C | 40°C |
| 2 | 35°C | 30°C | 25°C |
| 3 | 40°C | 25°C | 35°C |
| 4 | 40°C | 30°C | 35°C |



## 9.4 Analysis of BIB Designs

### Statistical Model for BIB Designs

The linear statistical model for a balanced incomplete block design is

$$y_{ij} = \mu + \tau_i + \rho_j + e_{ij} \tag{9.1}$$

$$i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, b$$

where $\mu$ is the general mean, $\tau_i$ is the fixed effect of the $i$th treatment, $\rho_j$ is the fixed effect of the $j$th block, and the $e_{ij}$ are independent, random experimental errors with mean 0 and variance $\sigma^2$.

Recall that there are $r$ replications of the $t$ treatments in $b$ incomplete blocks of $k$ experimental units. The total number of observations is $N = rt = bk$, with each treatment pair appearing together in $\lambda = r(k-1)/(t-1)$ blocks in the experiment.

The effects of treatments and blocks are not orthogonal in the incomplete block design because all treatments do not appear in each of the blocks. Therefore, the sum of squares partition for treatments computed in the manner of complete block designs will not be correct for the incomplete block designs, nor will the observed treatment means provide unbiased estimates of $\mu_i = \mu + \tau_i$. The parameter estimates and treatment sum of squares for the balanced incomplete block designs can be computed with relatively straightforward formulae. A derivation of the least squares estimates for the balanced incomplete block design is given in Appendix 9A.3.

### Sum of Squares Partitions for BIB Designs

The sum of squares partitions can be derived by considering alternative full and reduced models for the design. Solutions to the normal equations are obtained for the full model, $y_{ij} = \mu + \tau_i + \rho_j + e_{ij}$, with estimates, $\widehat{y}_{ij} = \widehat{\mu} + \widehat{\tau}_i + \widehat{\rho}_j$, to compute the experimental error sum of squares for the full model,

$$SSE_f = \sum_i \sum_j (y_{ij} - \widehat{y}_{ij})^2 = \sum_i \sum_j (y_{ij} - \widehat{\mu} - \widehat{\tau}_i - \widehat{\rho}_j)^2 \tag{9.2}$$

Solutions to the normal equations for the reduced model, $y_{ij} = \mu + \rho_j + e_{ij}$, with estimates, $\widehat{y}_{ij} = \widehat{\mu} + \widehat{\rho}_j$, are used to compute the experimental error sum of squares for the reduced model,

$$SSE_r = \sum_i \sum_j (y_{ij} - \widehat{y}_{ij})^2 = \sum_i \sum_j (y_{ij} - \widehat{\mu} - \widehat{\rho}_j)^2 \tag{9.3}$$

The difference, $SSE_r - SSE_f$, is the reduction in sum of squares as a result of including $\tau_i$ in the full model. It is the sum of squares due to treatments after block effects have been considered in the model. It is referred to as $SS$ Treatment(adjusted), implying that block effects are also considered when estimating the treatment effects in the full model. For balanced incomplete block designs the sum of squares for treatments adjusted, $SSE_r - SSE_f$, can be computed directly as

$$SST\text{(adjusted)} = \frac{k\sum_{i=1}^{t} Q_i^2}{\lambda t} \tag{9.4}$$

with $t-1$ degrees of freedom. The quantity $Q_i$ is an adjusted treatment total computed as

$$Q_i = y_{i.} - \frac{1}{k}B_i \tag{9.5}$$

where $B_i = \sum_j^b n_{ij}y_{.j}$ is the sum of all block totals that include the $i$th treatment, and $n_{ij} = 1$ if treatment $i$ appears in block $j$ and $n_{ij} = 0$ otherwise. This correction to the treatment total has the net effect of removing the block effects from the treatment total.

The sum of squares for blocks is derived from the reduced model with treatments ignored in the model as $SSB\text{(unadjusted)} = SS\text{ Total} - SSE_r$. Treatment effects are not considered when estimating the block effects, and the sum of squares is called an *unadjusted* sum of squares.

The additive partition of $SS$ Total is

$$SS\text{ Total} = SSB\text{(unadjusted)} + SST\text{(adjusted)} + SSE_f \tag{9.6}$$

The analysis of variance outline for the sum of squares partitions is shown in Table 9.1. Many computer programs are capable of computing the correct sum of squares partitions, least squares estimates of treatment means, contrasts among least squares means, and their standard errors for incomplete block designs.

**Table 9.1** Analysis of variance for a balanced incomplete block design

| *Source of Variation* | *Degrees of Freedom* | *Sum of Squares* | *Mean Square* |
|---|---|---|---|
| Total | $N-1$ | $\sum(y_{ij}-\bar{y}_{..})^2$ | |
| Blocks | $b-1$ | $k\sum(\bar{y}_{.j}-\bar{y}_{..})^2$ | $MSB$(unadj.) |
| Treatments | $t-1$ | $\frac{k\sum Q_i^2}{\lambda t}$ | $MST$(adj.) |
| Error | $N-t-b+1$ | By subtraction | $MSE$ |

**Example 9.2 Vinylation of Methyl Glucoside**

The addition of acetylene to methyl glucoside in the presence of a base under high pressure had been found to result in the production of several monovinyl ethers, a process known as *vinylation*. The monovinyl ethers are suitable for polymerization in many industrial applications.

***Research Objective*:** The chemists wanted to obtain more specific information about the effect of pressure on percent conversion of methyl glucoside to monovinyl isomers.



***Treatment Design*:** Based on previous work, pressures were selected within the range thought to produce maximum conversion. Five pressures were selected to estimate a response equation: 250, 325, 400, 475, and 550 psi.

***Experiment Design*:** Only three high-pressure chambers were available for one run of the experimental conditions. It was necessary to block on runs because there could be substantial run-to-run variation produced by new setups of the experiment in the high-pressure chambers. The chemists set up a balanced incomplete block design with ten blocks (runs) each with three experimental units (pressurized chambers). Three different pressures were used in each run. The resulting design had six replications of each pressure treatment.

The pressures used in each run and the percent conversions to monovinyl isomers are shown in Table 9.2. The additive sum of squares partition of methyl glucoside is shown in Table 9.3.

**Table 9.2** Percent conversion of methyl glucoside by acetylene under high pressure in a balanced incomplete block design

| | *Pressure (psi)* | | | | | |
|---|---|---|---|---|---|---|
| *Run* | *250* | *325* | *400* | *475* | *550* | $y_{.j}$ |
| 1 | 16 | 18 | – | 32 | – | 66 |
| 2 | 19 | – | – | 46 | 45 | 110 |
| 3 | – | 26 | 39 | – | 61 | 126 |
| 4 | – | – | 21 | 35 | 55 | 111 |
| 5 | – | 19 | – | 47 | 48 | 114 |
| 6 | 20 | – | 33 | 31 | – | 84 |
| 7 | 13 | 13 | 34 | – | – | 60 |
| 8 | 21 | – | 30 | – | 52 | 103 |
| 9 | 24 | 10 | – | – | 50 | 84 |
| 10 | – | 24 | 31 | 37 | – | 92 |
| $y_{i.}$ | 113 | 110 | 188 | 228 | 311 | 950 |
| $^*B_{i.}$ | 507 | 542 | 576 | 577 | 648 | |
| $^\dagger Q_i$ | – 56.0 | – 70.7 | – 4.0 | 35.7 | 95.0 | |

*Example: $B_1 = y_{.1} + y_{.2} + y_{.6} + y_{.7} + y_{.8} + y_{.9}$

†$Q_1 = y_{1.} - \frac{1}{3}B_1 = 113 - \frac{1}{3}(507) = -56.0$

*Source*: Drs. J. Berry and A. Deutschman, University of Arizona.

### Inferences for Treatment Means

The least squares estimates of the treatment means and their estimated standard errors are shown in Table 9.4. The least squares estimate for a treatment mean $\mu_i$ is $\widehat{\mu}_i = \widehat{\mu} + \widehat{\tau}_i$, where

**Table 9.3** Analysis of variance for percent conversion of methyl glucoside in a balanced incomplete block design

| *Source of Variation* | *Degrees of Freedom* | *Sum of Squares* | *Mean Squares* | *F* | *Pr > F* |
|---|---|---|---|---|---|
| Total | 29 | 5576.67 | | | |
| Blocks (unadj) | 9 | 1394.67 | 154.96 | | |
| Pressure(adj) | 4 | 3688.58 | 922.14 | 29.90 | .000 |
| Error | 16 | 493.42 | 30.84 | | |

$$\widehat{\mu} = \overline{y}_{..} \quad \text{and} \quad \widehat{\tau}_i = \frac{kQ_i}{\lambda t} \tag{9.7}$$

For example, from Table 9.2, $Q_1 = -56.00$ and $\widehat{\mu} = \overline{y}_{..} = 950/30 = 31.67$, so that

$$\widehat{\tau}_1 = \frac{kQ_1}{\lambda t} = \frac{3}{(3)(5)}(-56.00) = -11.20 \quad \text{and} \quad \widehat{\mu}_1 = 31.67 - 11.20 = 20.47$$

### *Standard Errors for Treatment Means*

The standard error for a treatment mean estimate is

$$s_{\widehat{\mu}_i} = \sqrt{\frac{MSE}{rt}\left(1 + \frac{kr(t-1)}{\lambda t}\right)} = \sqrt{\frac{30.84}{(6)(5)}\left(1 + \frac{(3)(6)(4)}{(3)(5)}\right)} = 2.44 \tag{9.8}$$

A 95% confidence interval estimate of a treatment mean in Table 9.4 is $\widehat{\mu}_i \pm t_{.025,16}(s_{\widehat{\mu}_i})$, where $t_{.025,16} = 2.120$. The standard error of the estimated difference between two treatment means, $\widehat{\mu}_i - \widehat{\mu}_j$, is

$$s_{(\widehat{\mu}_i - \widehat{\mu}_j)} = \sqrt{\frac{2kMSE}{\lambda t}} = \sqrt{\frac{(2)(3)30.84}{(3)(5)}} = 3.51 \tag{9.9}$$

**Table 9.4** Least squares estimates of pressure means for percent conversion of methyl glucoside in a balanced incomplete block design

| *Pressure (psi)* | $\widehat{\mu}_i$ | $s_{\widehat{\mu}_i}$ |
|---|---|---|
| 250 | 20.47 | 2.44 |
| 325 | 17.53 | 2.44 |
| 400 | 30.87 | 2.44 |
| 475 | 38.80 | 2.44 |
| 550 | 50.67 | 2.44 |



### Tests of Hypotheses About Treatment Means

The $F_0$ statistic to test the null hypothesis of no differences among the treatment means is

$$F_0 = \frac{MST(\text{adj.})}{MSE} = \frac{922.14}{30.84} = 29.90 \tag{9.10}$$

with $Pr > F = .000$ (Table 9.3). The critical value at the .05 level of significance is $F_{.05,4,16} = 3.01$. There are significant differences among the pressures with respect to conversion of methyl glucoside to vinylation products.

### Contrasts Among Treatment Means

Treatment contrasts are estimated with least squares estimates of the treatment means $\widehat{\mu}_i$ as

$$c = \sum_{i=1}^{t} d_i \widehat{\mu}_i \tag{9.11}$$

with standard error estimate

$$s_c = \sqrt{\frac{kMSE}{\lambda t}\left(\sum_{i=1}^{t} d_i^2\right)} \tag{9.12}$$

The statistic $t_0 = c/s_c$ can be used to test the null hypothesis $H_0$: $C = 0$ with a critical value based on the Student $t$ statistic with $N - t - b + 1$ degrees of freedom.

The 1 degree of freedom sum of squares for the contrast can be computed with the least squares means $\widehat{\mu}_i$ as

$$SSC = \frac{\lambda t(\Sigma\, d_i \widehat{\mu}_i)^2}{k\, \Sigma\, d_i^2} \tag{9.13}$$

The statistic $F_0 = SSC/MSE$ is used to test the null hypothesis $H_0$: $C = 0$ with critical value $F_{\alpha,1,(N-t-b+1)}$.

The pressure treatment for this experiment is a quantitative factor with five levels. The regression of percent conversion of methyl glucoside on pressure using orthogonal polynomial contrasts for pressure will describe the effect of pressure on conversion rate. The regression analysis is left as an exercise at the end of the chapter.

### Recovering Treatment Information from Block Comparisons

The analyses for incomplete block designs illustrated thus far estimate treatment effects based on treatment information contained within the blocks; this is referred to as *intrablock* analysis. The incomplete block designs are nonorthogonal because not all treatments appear in all blocks, and comparisons among the blocks contain some information about treatment comparisons. Yates (1940a) showed this *interblock* information can be recovered with an interblock analysis and combined with the information from the intrablock analysis.

### *Block Contrasts Contain Treatment Contrasts*

The information on treatment comparisons contained in a comparison between blocks can be illustrated simply with the first two runs of the chemistry experiment in Example 9.2. The data for percent conversion of methyl glucoside in the first two runs of the experiment are

| | *Presssure (psi)* | | | | | |
|---|---|---|---|---|---|---|
| *Run* | *250* | *325* | *400* | *475* | *550* | *Mean* |
| 1 | 16 | 18 | – | 32 | – | 22 |
| 2 | 19 | – | – | 46 | 45 | 37 |

A contrast between the two runs means, 37 and 22, is also a contrast between two of the pressure treatments, 550 psi and 325 psi. Similar treatment comparisons are contained in all of the block comparisons. The objective is to recover this between-block, or interblock, information about treatments and combine it with the within-block, or intrablock, information about treatments.

The subject of interblock information recovery is mentioned here only to indicate the availability of the method. A thorough treatment on the recovery of interblock information can be found in Kempthorne (1952), Cochran and Cox (1957), John (1971), John (1987), and John and Williams (1995).

Information from the interblock analysis is incorporated into the intrablock analysis with an estimator of the treatment effects that combines the intrablock and interblock estimators of the treatment effects.

If blocking has been very effective in reducing experimental error, the interblock estimate contributes only a small amount of information to the combined estimate. On the other hand, if the block effects are small the information recovered from the interblock estimate can be substantial. If blocking is not effective, then the analysis with recovery of interblock information virtually reduces to the ordinary analysis with no block adjustments.

## 9.5 Row–Column Designs for Two Blocking Criteria

When the need exists to control variation with more than one blocking criterion the Latin square design is a complete block design used to control variation among the experimental units with two blocking factors. Latin square designs may be impractical in some situations since the number of experimental units required by the design, $N = t^2$, can exceed the constraints of the experimental material or treatment numbers can exceed available block sizes.



Row–column designs can be used with either rows or columns or both rows and columns as incomplete blocks when two blocking criteria are required for the experiment. The designs are arranged in $p$ rows and $q$ columns of experimental units. Consider the classical experiment to test four automobile tire treads on the four positions of four automobiles in a Latin square design. Suppose the research team wanted to evaluate $t = 7$ tire treatments. The requirement to control variation due to tire position and automobile is still necessary. However, the automobiles only have four positions on which to test the seven treatments. The row–column design with an incomplete set of treatments in each column shown in Display 9.2 can be used for the experiment.

**Display 9.2 A 4 × 7 Row–Column Balanced Incomplete Block Design**

| | *Automobile* | | | | | | |
|---|---|---|---|---|---|---|---|
| *Position* | *(1)* | *(2)* | *(3)* | *(4)* | *(5)* | *(6)* | *(7)* |
| (1) | 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| (2) | 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| (3) | 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| (4) | 7 | 1 | 2 | 3 | 4 | 5 | 6 |

The automobiles are used as incomplete blocks with four treatments evaluated on the four tire positions. The positions are complete blocks since each treatment is evaluated at each of the positions. Upon close inspection it can be seen that each treatment pair occurs on the same automobile two times somewhere in the experiment. Therefore, the incomplete block design on automobiles is balanced. The design is naturally balanced with respect to the positions because they constitute complete blocks. The example is clearly a case where an incomplete block design was necessary because there was an insufficient number of positions available to test all treatments at one time.

### Row Orthogonal Designs Have a Complete Replication in Each Row

Given the row–column design in Display 9.2 is a complete block design for rows and a balanced incomplete block design for columns the design is referred to as a row orthogonal design (John, 1987). Since each treatment occurs in each row, the treatments are orthogonal to the rows.

Youden (1937, 1940) developed incomplete Latin square arrangements, now known as *Youden squares*, by omitting two or more rows from the Latin square design. The design parameters are $t = b$, $r = k$, and $\lambda = k(k-1)/(t-1)$. Plans of some Youden squares for small experiments are shown in Appendix 9A.2. Additional plans can be found in Cochran and Cox (1957) and Petersen (1985).

The row orthogonal design in Display 9.2 is a Youden square that has $r = k = 4$ rows as replications, $b = 7$ columns, and $t = 7$ treatments, with $\lambda = 2$

for the incomplete column blocks. The columns are incomplete blocks of $k = r = 4$ units, and the rows are complete blocks containing each of the $t = 7$ treatments.

Randomization in row–column designs is accomplished in the same manner as that described for Latin square designs in Chapter 8. There is a separate permutation of each of the row and column groups of treatments to the actual blocks, and the treatments are randomly assigned to the treatment labels of the design.

### The Analysis Outline for Row–Column Designs

The linear model for the row–column design is

$$y_{ijm} = \mu + \tau_i + \rho_j + \gamma_m + e_{ijm} \tag{9.14}$$

$$i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, k \quad m = 1, 2, \ldots, b$$

where $\mu$ is the general mean, $\tau_i$ is the treatment effect, $\rho_j$ is the row effect, $\gamma_m$ is the column effect, and $e_{ijm}$ is the random experimental error.

Each row contains a complete replication of all of the treatments, and the treatments are orthogonal to the rows. The columns are also orthogonal to the rows. The treatment totals are adjusted only for the incomplete column blocks to provide unbiased estimates of treatment means and a valid $F$ test for treatment effects. The intrablock analysis of variance outline is shown in Table 9.5. The analysis of variance for the row orthogonal designs only differs from the balanced incomplete block designs by the addition of a sum of squares partition for the rows. All other aspects of the analysis remain the same.

**Table 9.5** Intrablock analysis for a row–column balanced incomplete block design with treatments orthogonal to rows

| *Source of Variation* | *Degrees of Freedom* | *Sum of Squares* |
|---|---|---|
| Total | $N-1$ | $\sum(y_{ijm} - \bar{y}_{...})^2$ |
| Rows (replications) | $k-1$ | $t\sum(\bar{y}_{.j.} - \bar{y}_{...})^2$ |
| Columns (unadjusted) | $b-1$ | $k\sum(\bar{y}_{..m} - \bar{y}_{...})^2$ |
| Treatments (adjusted)* | $t-1$ | $k\sum Q_i^2/\lambda t$ |
| Error | $(t-1)(k-1)-(b-1)$ | By subtraction |

*$Q_i = y_{i..} - (B_i/k)$ where $B_i$ is the sum of column block totals that include treatment $i$.

## 9.6 Reduce Experiment Size with Partially Balanced (PBIB) Designs

Balanced designs cannot be constructed for every experimental situation requiring incomplete blocks. In some cases the required number of replications may become prohibitive. Frequently, partially balanced designs requiring much less replication can be constructed. The minimum number of replications required for the balanced design is $r = \lambda(t-1)/(k-1)$. Suppose an experiment with $t = 6$ treatments requires blocks of size $k = 4$. The balanced design shown in Appendix 9A.1 requires $r = 10$ replications or $rt = 60$ experimental units. It is completely possible that 60 experimental units are not available or that the cost of the experiment with 60 units is prohibitive.



### Unequal Occurrences of Treatment Pairs

The partially balanced incomplete block design was introduced by Bose and Nair (1939). The partially balanced design has some treatment pairs occurring in more blocks than other treatment pairs, and consequently some treatment comparisons will be made with greater precision than others. It is more straightforward to use a balanced design that provides equal precision for all comparisons between treatments. However, if resources are limited and sufficient replication is possible, the partially balanced design is an attractive alternative to the balanced design that requires excessive numbers of experimental units.

Consider the partially balanced incomplete block design for six treatments in blocks of four units shown in Display 9.3. The balanced design in Appendix 9A.1 requires ten replications for balance with $\lambda = 6$. The partially balanced design shown in Display 9.3 has two replications and requires twelve experimental units in three blocks of four experimental units.

**Display 9.3 A Partially Balanced Incomplete Block Design with Six Treatments in Blocks of Four Units**

| | | | | |
|---|---|---|---|---|
| Block 1 | 1 | 4 | 2 | 5 |
| Block 2 | 2 | 5 | 3 | 6 |
| Block 3 | 3 | 6 | 1 | 4 |

Some pairs of treatments occur together in two blocks, whereas other pairs occur together in only one block. Treatment pairs $(1,4), (2,5)$, and $(3,6)$ occur together in two blocks. All other treatment pairs occur together in only one block. Those treatment pairs occurring together in two blocks will be compared with somewhat greater precision than those pairs that occur together in only one block. The differing precisions for treatment comparisons is the sacrifice made for the smaller experiment. However, the difference in precision is not so great as to inhibit the use of partially balanced designs.

Increasing the replication number can increase the precision on the treatment comparisons. Another repetition of the same experiment will provide four replications if the two replications provided by the initial experiment are insufficient. If four replications are sufficient, there is still a gain with the partially balanced design in terms of reduced costs over the fully balanced design.

### Associate Classes for Treatment Pair Occurrences

Each treatment is a member of two or more associate classes in a partially balanced design. An *associate class* is a group of treatments wherein each treatment pair occurs together in $\lambda_i$ blocks. Treatment pairs that occur together in $\lambda_i$ blocks are known as $i$th associates.

The design in Display 9.3 has two associate classes. The treatment pairs $(1,4), (2,5)$, and $(3,6)$ are first associates with $\lambda_1 = 2$. Each pair occurs together in two blocks. Each treatment has $n_1 = 1$ first associates. The remaining treatment pairs are second associates with $\lambda_2 = 1$. There are $n_2 = 4$ second associates for each treatment. For example, the second associates of treatment 1 are treatments 2, 3, 5, and 6. They occur with treatment 1 somewhere in one block of the design. The complete sets of associates are shown in Display 9.4.

**Display 9.4 First and Second Associates for Six Treatments in the Partially Balanced Design of Display 9.3**

| | *Associates* | |
|---|---|---|
| *Treatment* | *First* | *Second* |
| 1 | 4 | 2, 3, 5, 6 |
| 2 | 5 | 1, 3, 4, 6 |
| 3 | 6 | 1, 2, 4, 5 |
| 4 | 1 | 2, 3, 5, 6 |
| 5 | 2 | 1, 3, 4, 6 |
| 6 | 3 | 1, 2, 4, 5 |

A catalog for some of the major groups of partially balanced designs with two associate classes can be found in Bose, Clatworthy, and Shrikhande (1954) and Clatworthy (1973).

### Notes on the Analysis of PBIB Designs

One advantage of the balanced designs over the partially balanced designs is that the manual calculations for the analysis of variance are somewhat simpler. Before the advent of modern-day statistical programs it was imperative that realistic manual computational formulae be available for an analysis of the data. Now it is possible to compute the appropriate sum of squares partitions for the analysis of variance as well as unbiased estimates of treatment means and their standard errors for most designs with available statistical programs. Therefore, the partially balanced design



is a realistic alternative provided there are reasonably precise comparisons between all treatment pairs.

#### *Sum of Squares Partitions for PBIB Designs*

The linear model for the partially balanced incomplete block design is the same as that for the balanced incomplete block design shown in Equation (9.1). Since the treatments are not orthogonal to blocks, the orthogonal sum of squares partitions are again derived by fitting the full model, $y_{ij} = \mu + \tau_i + \rho_j + e_{ij}$, to obtain $SSE_f$ and the alternative reduced model without treatment effects, $y_{ij} = \mu + \rho_j + e_{ij}$, to obtain $SSE_r$.

The sum of squares for treatments adjusted for blocks is derived as $SS$ Treatment (adjusted) $= SSE_r - SSE_f$. The block sum of squares unadjusted for treatments is the same as that shown in Table 9.1 for the balanced design.

The formulae to compute least squares estimates of treatment effects, treatment means, and the adjusted treatment sum of squares are not so straightforward as those for the balanced designs. Because there is more than one associate class for each of the treatments, more complex adjustments must be made to the treatment totals. Fortunately, many of the statistical programs available can perform the computations. Thus, it is not necessary to utilize the cumbersome manual calculation formulae for a thorough analysis of the data. Details of manual calculations can be found in Cochran and Cox (1957). Detailed derivations can be found in John (1971) and John (1987).

## 9.7 Efficiency of Incomplete Block Designs

The efficiency of one design relative to another is measured by comparing the variance for the estimates of treatment mean differences in the two designs. For example, the variance of the difference between two treatment means for the *randomized complete block* (RCB) design is $2\sigma^2_{rcb}/r$. Is that variance less than its counterpart from the completely randomized design $2\sigma^2_{cr}/r$? The relative efficiency of the RCB design to the completely randomized design can be determined because an estimate of $\sigma^2$ for the latter can be obtained from the data in the RCB design. It is then possible to evaluate the effectiveness of blocking to reduce experimental error variance.

No such luxury exists if we want to determine the efficiency of a balanced incomplete block (BIB) design relative to the RCB design. It is not possible to compute an estimate of $\sigma^2$ for the RCB design from the data in the BIB design. We would like to determine whether the smaller block size of the BIB design resulted in a smaller experimental error variance. The ratio of variances for a difference between two treatment means is still used to compare the BIB design with the RCB design. The difference is that it measures only the *potential* efficiency of the BIB design since we are unable to estimate $\sigma^2$ for the RCB design.

### The Efficiency Factor for Incomplete Block Designs

The variance of the difference between two treatment means in the BIB design is $2k\sigma^2_{bib}/\lambda t$. If the BIB design and the RCB design have the *same number of treatments and replications* the efficiency of the BIB design relative to the RCB design is the ratio of the variances

$$\text{Efficiency} = \frac{(2\sigma^2_{rcb}/r)}{(2k\sigma^2_{bib}/\lambda t)} = \frac{\sigma^2_{rcb}}{\sigma^2_{bib}} \cdot \frac{\lambda t}{rk} \tag{9.15}$$

The quantity $E = \lambda t/rk$ is called the *efficiency factor* for the balanced incomplete block design. It provides an indication of the loss caused by using an incomplete block design without reducing $\sigma^2$. The BIB design is more precise for the comparison of two treatment means than the RCB design if

$$\frac{\sigma^2_{bib}}{\sigma^2_{rcb}} < \frac{\lambda t}{rk} \tag{9.16}$$

The efficiency factor is a lower limit to the efficiency of the BIB design relative to the RCB design for an experiment with the same number of replications and the same error variance $\sigma^2$. Given a BIB design with $t = 4, r = 3, k = 3$, and $\lambda = 2$ and an RCB design with $t = 4$ and $r = 3$, both designs require the same number of experimental units but they have different blocking arrangements. The BIB design is more precise than the RCB design if

$$\frac{\sigma^2_{bib}}{\sigma^2_{rcb}} < \frac{2(4)}{3(3)} = 0.89$$

In other words, $\sigma^2_{bib}$ would have to be about 11% smaller than $\sigma^2_{rcb}$ for the BIB design to be as precise as the RCB design with the same number of replications.

The intent of an incomplete block design is to reduce the error variance and increase the precision of the comparisons between treatment means. The goal would be to reduce $\sigma^2$ for the BIB design so that the inequality shown in Equation (9.16) would be achieved. A successful incomplete block design will reduce the error variance, and $\sigma^2_{bib}$ will be smaller than $\sigma^2_{rcb}$. Some elements that enter into a successful blocking strategy are considered in the next chapter.

## EXERCISES FOR CHAPTER 9

1. A horticulturalist studied the germination of tomato seed with four different temperatures (25°C, 30°C, 35°C, and 40°C) in a balanced incomplete block design because there were only two growth chambers available for the study. The experiment was conducted in a balanced incomplete block design. Each run of the experiment was an incomplete block consisting of the two growth chambers as the experimental units ($k = 2$). Two experimental temperatures were randomly assigned to the chambers for each run. The data that follow are germination rates of the tomato seed.



| *Run* | *25 °C* | *30 °C* | *35 °C* | *40 °C* |
|---|---|---|---|---|
| 1 | 24.65 | — | — | 1.34 |
| 2 | — | 24.38 | — | 2.24 |
| 3 | 29.17 | 21.25 | — | — |
| 4 | — | — | 5.90 | 1.83 |
| 5 | 28.90 | — | 18.27 | — |
| 6 | — | 25.53 | 8.42 | — |

*Source*: Dr. J. Coons, Department of Botany, Eastern Illinois University.

a. How many times did each treatment pair occur together in the same block?
b. What is the efficiency factor for this design?
c. Write a linear model for the experiment, explain the terms, compute the intrablock analysis of variance, and test the null hypothesis for temperature effects.
d. Compute the least squares estimates of the temperature means and their standard error.
e. Compute the standard error of the difference between two least squares estimates of temperature means.
f. Compute the 1 degree of freedom sum of squares partitions for linear and quadratic orthogonal polynomial regression, and test their null hypotheses.
g. Are the deviations from quadratic regression significant?

2. A company is developing an insulating compound. An accelerated life testing procedure is conducted to determine the breakdown time in minutes after subjection to elevated voltages. Four voltages could be tested on a given run of the test. A balanced incomplete block design was used to test the compound with $t = 7$ voltages in $b = 7$ runs of $k = 4$ tests. The minutes to compound breakdown were recorded for each test.

| | *Voltage (kv)* | | | | | | |
|---|---|---|---|---|---|---|---|
| *Block* | *24* | *28* | *32* | *36* | *40* | *44* | *48* |
| 1 | — | — | 38.19 | — | 5.44 | 1.96 | 0.55 |
| 2 | 220.22 | — | — | 7.66 | — | 2.54 | 0.67 |
| 3 | 270.85 | 200.67 | — | — | 6.24 | — | 0.76 |
| 4 | 360.14 | 170.52 | 45.43 | — | — | 3.22 | — |
| 5 | — | 220.12 | 56.74 | 9.32 | — | — | 0.61 |
| 6 | 300.66 | — | 55.34 | 10.41 | 7.19 | — | — |
| 7 | — | 190.78 | — | 8.74 | 6.92 | 2.21 | — |

a. How many times did each treatment pair occur together in the same block?
b. What is the efficiency factor for this design?
c. Write a linear model for the experiment, explain the terms, and compute the intrablock analysis of variance.
d. Compute the least squares residuals for each observation, and analyze the residuals according to the procedures discussed in Chapter 4. A typical transformation utilized for data of this nature is the natural logarithm. Does that seem appropriate here?

e. Transform the data, compute the additive sum of squares partitions, and summarize in an analysis of variance table. Describe how the transformation has changed the nature of the residuals.
f. Compute the least squares estimates of the voltage means and their standard error.
g. Compute the standard error of the difference between two least squares estimates of voltage means.
h. Compute the 1 degree of freedom sum of squares partition for linear regression of time to breakdown (or a suitable transformation) on voltage, and test the null hypotheses.
i. Are the deviations from linear regression significant?

3. A study was conducted to evaluate a method to measure traffic delay at urban street intersections in seven cities. Seven types of intersections were chosen for the study based on their geometry and traffic light configurations. Four of the intersection types were measured in each city in an incomplete block design. Four observers were used to measure traffic delay, one at each of the intersections for a specified period of time during peak traffic loads. The design was laid out as a Youden square with $t = 7$ intersection type treatments. The row–column blocking consisted of $k = 4$ observers as rows and $b = 7$ cities as columns. The order in which the cities were visited and the sequence of intersection type treatments each of the observers measured were randomized according to the Latin square procedure. The intersection types were randomly assigned to the treatment labels of the basic Youden square and are shown in parentheses in the data table for time in queue (seconds per vehicle).

| | *City* | | | | | | |
|---|---|---|---|---|---|---|---|
| *Observer* | *1* | *2* | *3* | *4* | *5* | *6* | *7* |
| 1 | (4) | (1) | (3) | (6) | (2) | (7) | (5) |
| | 45.8 | 66.4 | 27.0 | 92.6 | 32.7 | 34.6 | 44.1 |
| 2 | (2) | (6) | (1) | (4) | (7) | (5) | (3) |
| | 28.6 | 94.6 | 64.6 | 39.9 | 34.5 | 45.7 | 23.7 |
| 3 | (7) | (4) | (6) | (2) | (5) | (3) | (1) |
| | 32.3 | 40.7 | 82.6 | 29.3 | 47.6 | 31.2 | 74.7 |
| 4 | (3) | (7) | (2) | (5) | (1) | (6) | (4) |
| | 28.0 | 31.1 | 25.5 | 41.9 | 68.7 | 68.1 | 38.7 |

a. Write a linear model for this study, describe the terms, and conduct the intrablock analysis of variance for this experiment.
b. Compute the standard error of the difference between least squares means of two intersection means.
c. What is the efficiency factor for this design?
d. Use the Multiple Comparisons with the Best procedure to select the set of intersection types with the minimum delay.

4. (*Note*: This exercise presumes you have available a program to compute the analysis of variance, least squares estimates of treatment means with their standard errors, and contrasts with their standard errors.) A study was conducted in a partially balanced incomplete block design to evaluate $t = 9$ feed rations on nitrogen balance in ruminants. The study required expensive digestion stalls and equipment, and only three such stalls were available for the study. The design consisted of $r = 3$ replications of the 9 treatments in $b = 9$ blocks of $k = 3$ animals. There were $n_1 = 6$ first associates for each treatment with $\lambda_1 = 1$ and $n_2 = 2$ second associates for each treatment with $\lambda_2 = 0$. The observed responses are shown in the table with treatment numbers in parentheses.



| *Block* | | | |
|---|---|---|---|
| 1 | 33.72 (1) | 37.80 (2) | 42.25 (3) |
| 2 | 38.58 (1) | 45.39 (4) | 47.75 (6) |
| 3 | 34.55 (1) | 34.82 (5) | 38.29 (7) |
| 4 | 42.95 (2) | 45.35 (4) | 48.84 (9) |
| 5 | 45.12 (2) | 36.36 (5) | 40.58 (8) |
| 6 | 43.04 (3) | 45.22 (6) | 37.26 (8) |
| 7 | 40.64 (3) | 30.49 (7) | 36.34 (9) |
| 8 | 38.53 (4) | 34.58 (7) | 40.81 (8) |
| 9 | 36.40 (5) | 33.28 (6) | 38.46 (9) |

*Source:* J. L. Gill (1978), *Design and analysis of experiments in the animal and medical sciences*, Vol 2. Ames, Iowa: Iowa State University Press.

a. Make a table of first and second associates for each treatment.
b. Write a linear model for the experiment, and explain the terms.
c. Use your available computer program to compute the analysis of variance to obtain the adjusted treatment mean square and the experimental error variance. Test the null hypothesis for rations.
d. Compute the least squares estimates of the ration means and their standard errors.
e. Compute the standard error of the difference between two least squares estimates of ration means for first associates and second associates.
f. Frequently, a weighted average standard error is utilized for all comparisons among two treatment means in a partially balanced design. With $n_1 = 6$ first associates and $n_2 = 2$ second associates, the weighted average variance of the difference between two treatment means for this study is

$$\text{average variance} = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$$

where $s_1^2$ and $s_2^2$ are the estimated variances of the differences between first and second associates, respectively. Compute a weighted average standard error of the difference for this experiment.

5. An incomplete block design consists of the following arrangement of blocks (1, 2, 3, 4, 5) and treatments (A, B, C, D, E).

| *Block* | | |
|---|---|---|
| | 1 | (B, C, D, E) |
| | 2 | (A, B, D, E) |
| | 3 | (A, C, D, E) |
| | 4 | (A, B, C, D) |
| | 5 | (A, B, C, E) |

a. What are the design parameters $t, r, k$, and $b$?
b. Verify that the design is balanced.

6. An incomplete block design consists of the following arrangement of blocks 1 through 6 and treatments (A, B, C, D, E, F).

| | | | | | |
|---|---|---|---|---|---|
| *Block* | 1 | (A, B, C) | *Block* | 4 | (A, B, D) |
| | 2 | (C, D, E) | | 5 | (C, D, F) |
| | 3 | (B, E, F) | | 6 | (A, E, F) |

a. What are the design parameters $t, r, k$, and $b$?
b. Is the design balanced? Explain.

7. Suppose you are to test $t = 6$ automobile fuels, and you have $k = 3$ engines available for the tests. It is therefore necessary to use an incomplete block design for the tests with runs of the three engines as blocks. Find the appropriate balanced incomplete block design in Appendix 9A.1 to conduct the experiment. Randomize the treatment groups to blocks of runs and automobile fuel treatments to the engines.

8. A paired comparisons study is to be conducted to evaluate campsites in a national park. There are $t = 10$ campsite designs, which range from primitive to full-facility sites. Visitors to the national park will be selected at random during the month of June and shown photographs of $k = 2$ of the ten campsites. They are to indicate their preference for one of the two campsites based on the photographs. Construct a balanced incomplete block design for the study. How many visitors will be required for the study? How many replications of each campsite will there be in the study? How many times will each pair of campsites be viewed by a visitor? How would the study design change if each visitor was asked to view $k = 3$ campsites?

## 9A.1 Appendix: Selected Balanced Incomplete Block Designs

**Plan 9A.1** $t = 5, k = 3, r = 6, b = 10, \lambda = 3, E = 0.83$
(1,2,3), (1,2,5), (1,4,5), (2,3,4), (3,4,5) reps 1,2,3
(1,2,4), (1,3,4), (1,3,5), (2,3,5), (2,4,5) reps 4,5,6

**Plan 9A.2** $t = 6, k = 3, r = 5, b = 10, \lambda = 2, E = 0.80$
(1,2,5), (1,2,6), (1,3,4), (1,3,6), (1,4,5),
(2,3,4), (2,3,5), (2,4,6), (3,5,6), (4,5,6)

**Plan 9A.3** $t = 6, k = 4, r = 10, b = 15, \lambda = 6, E = 0.90$
(1,2,3,4), (1,4,5,6), (2,3,5,6) reps 1,2
(1,2,3,5), (1,2,4,6), (3,4,5,6) reps 3,4
(1,2,3,6), (1,3,4,5), (2,4,5,6) reps 5,6
(1,2,4,5), (1,3,5,6), (2,3,4,6) reps 7,8
(1,2,5,6), (1,3,4,6), (2,3,4,5) reps 9,10



**Plan 9A.4** $t = 7, k = 3, r = 3, b = 7, \lambda = 1, E = 0.78$
(1,2,4), (2,3,5), (3,4,6), (4,5,7), (1,5,6),
(2,6,7), (1,3,7)

**Plan 9A.5** $t = 7, k = 4, r = 4, b = 7, \lambda = 2, E = 0.88$
Replace each block in Plan 9A.4 by a block containing the remaining treatments.

**Plan 9A.6** $t = 8, k = 4, r = 7, b = 14, \lambda = 3, E = 0.86$
(1,2,3,4), (5,6,7,8) rep 1 (1,2,7,8), (3,4,5,6) rep 2
(1,3,6,8), (2,4,5,7) rep 3 (1,4,6,7), (2,3,5,8) rep 4
(1,2,5,6), (3,4,7,8) rep 5 (1,3,5,7), (2,4,6,8) rep 6
(1,4,5,8), (2,3,6,7) rep 7

**Plan 9A.7** $t = 9, k = 3, r = 4, b = 12, \lambda = 1, E = 0.75$
(1,2,3), (1,4,7), (1,5,9), (1,6,8), (2,4,9), (2,5,8)
(2,6,7), (3,4,8), (3,5,7), (3,6,9), (4,5,6), (7,8,9)

**Plan 9A.8** $t = 9, k = 4, r = 8, b = 18, \lambda = 3, E = 0.84$
(1,2,3,4), (1,3,8,9), (1,4,6,7), (1,5,7,8), (2,3,6,7),
(2,4,5,8), (2,6,8,9), (3,5,7,9), (4,5,6,9), reps 1,2,3,4
(1,2,4,9), (1,2,5,7), (1,3,6,8), (1,5,6,9), (2,3,5,6),
(2,7,8,9), (3,4,5,8), (3,4,7,9), (4,6,7,8), reps 5,6,7,8

**Plan 9A.9** $t = 9, k = 5, r = 10, b = 18, \lambda = 5, E = 0.90$
Replace each block of Plan 9A.8 by a block containing the remaining treatments.

**Plan 9A.10** $t = 9, k = 6, r = 8, b = 12, \lambda = 5, E = 0.94$
Replace each block of Plan 9A.7 by a block containing the remaining treatments.

**Plan 9A.11** $t = 10, k = 3, r = 9, b = 30, \lambda = 2, E = 0.74$
(1,2,3), (1,4,6), (1,7,9), (2,5,8), (2,8,10)
(3,4,7), (3,9,10), (4,6,9), (5,6,10), (5,7,8) reps 1,2,3
(1,2,4), (1,5,7), (1,8,10), (2,3,6), (2,5,9),
(3,4,8), (3,7,10), (4,5,9), (6,7,10), (6,8,9) reps 4,5,6
(1,3,5), (1,6,8), (1,9,10), (2,4,10), (2,6,7),
(2,7,9), (3,5,6), (3,8,9), (4,5,10), (4,7,8) reps 7,8,9

**Plan 9A.12** $t = 10, k = 4, r = 6, b = 15, \lambda = 2, E = 0.83$
(1,2,3,4), (1,2,5,6), (1,3,7,8), (1,4,9,10), (1,5,7,9),
(1,6,8,10), (2,3,6,9), (2,4,7,10), (2,5,8,10), (2,7,8,9),
(3,5,9,10), (3,6,7,10), (3,4,5,8), (4,5,6,7), (4,6,8,9)

**Plan 9A.13** $t = 10, k = 5, r = 9, b = 18, \lambda = 4, E = 0.89$
(1,2,3,4,5), (1,2,3,6,7), (1,2,4,6,9), (1,2,5,7,8), (1,3,6,8,9), (1,3,7,8,10),

(1,4,5,6,10), (1,4,8,9,10), (1,5,7,9,10), (2,3,4,8,10), (2,3,5,9,10), (2,4,7,8,9), (2,5,6,8,10), (2,6,7,9,10), (3,4,6,7,10), (3,4,5,7,9), (3,5,6,8,9), (4,5,6,7,8)

**Plan 9A.14** $t = 10, k = 6, r = 9, b = 15, \lambda = 5, E = 0.93$
Replace each block in Plan 9A.12 by a block containing the remaining treatments.

**Plan 9A.15** $t = 11, k = 5, r = 5, b = 11, \lambda = 2, E = 0.88$
(1,2,3,5,8), (1,2,4,7,11), (1,3,6,10,11), (1,4,8,9,10), (1,5,6,7,9), (2,3,4,6,9), (2,5,9,10,11), (2,6,7,8,10), (3,4,5,7,10), (3,7,8,9,11), (4,5,6,8,11)

**Plan 9A.16** $t = 11, k = 6, r = 6, b = 11, \lambda = 3, E = 0.92$
Replace each block in Plan 9A.15 by a block containing the remaining treatments.

## 9A.2 Appendix: Selected Incomplete Latin Square Designs

**Table 9A** Plans for incomplete Latin square designs derived from complete Latin squares

| $t$ | $k$ | $r$ | $b$ | $\lambda$ | $E$ | *Plan* |
|---|---|---|---|---|---|---|
| 4 | 3 | 3 | 4 | 2 | .89 | * |
| | 3 | 6 | 8 | 4 | .89 | ** |
| | 3 | 9 | 12 | 6 | .89 | ** |
| 5 | 4 | 4 | 5 | 3 | .94 | * |
| | 4 | 8 | 10 | 6 | .94 | ** |
| 6 | 5 | 5 | 6 | 4 | .96 | * |
| | 5 | 10 | 12 | 8 | .96 | ** |
| 7 | 6 | 6 | 7 | 5 | .97 | * |
| 8 | 7 | 7 | 8 | 6 | .98 | * |
| 9 | 8 | 8 | 9 | 7 | .98 | * |
| 10 | 9 | 9 | 10 | 8 | .99 | * |
| 11 | 10 | 10 | 11 | 9 | .99 | * |

* Constructed from a $t \times t$ Latin square by omission of the last column
** By repetition of the plan for $r = t - 1$, which is constructed by taking a $t \times t$ Latin square and omitting the last column

**Plans for other incomplete Latin square designs:**

**Plan 9B.1** $t = 5,\ k = 2,\ r = 4,\ b = 10,\ \lambda = 1,\ E = .63$

*Reps I and II*

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 5 | 4 | 1 | 3 |

*Reps III and IV*

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 2 | 5 | 1 |



**Plan 9B.2** $t = 5, k = 3, r = 6, b = 10, \lambda = 3, E = .83$

*Reps I, II, and III*

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 1 | 4 | 5 | 3 |
| 3 | 5 | 2 | 1 | 4 |

*Reps IV, V, and VI*

| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 1 |
| 4 | 5 | 1 | 2 | 3 |

**Plan 9B.3** $t = 7,\ k = 2,\ r = 6,\ b = 21,\ \lambda = 1,\ E = .58$

*Reps I and II*

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 6 | 4 | 7 | 1 | 5 | 3 |

*Reps III and IV*

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |

*Reps V and VI*

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 4 | 3 | 6 | 5 | 2 | 7 | 1 |

**Plan 9B.4** $t = 7,\ k = 3,\ r = 3,\ b = 7,\ \lambda = 1,\ E = .78$

| | | | | | | |
|---|---|---|---|---|---|---|
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |

**Plan 9B.5** $t = 7,\ k = 4,\ r = 4,\ b = 7,\ \lambda = 2,\ E = .88$

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 6 | 7 | 1 | 2 |
| 5 | 6 | 7 | 1 | 2 | 3 | 4 |
| 6 | 7 | 1 | 2 | 3 | 4 | 5 |
| 7 | 1 | 2 | 3 | 4 | 5 | 6 |

**Plan 9B.6** $t = 9,\ k = 2,\ r = 8,\ b = 36,\ \lambda = 1,\ E = .56$

*Reps I and II*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 8 | 4 | 7 | 6 | 1 | 3 | 9 | 5 |

*Reps V and VI*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 6 | 2 | 5 | 7 | 8 | 9 | 1 | 3 |

*Reps III and IV*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 5 | 6 | 9 | 8 | 7 | 1 | 4 | 2 |

*Reps VII and VIII*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 4 | 8 | 6 | 3 | 9 | 2 | 7 | 1 |

**Plan 9B.7** $t = 9,\ k = 4,\ r = 8,\ b = 18,\ \lambda = 3,\ E = .84$

*Reps I, II, III, and IV*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 6 | 8 | 1 | 7 | 9 | 3 | 2 | 5 |
| 6 | 8 | 9 | 3 | 1 | 4 | 2 | 5 | 7 |
| 7 | 9 | 1 | 2 | 8 | 5 | 6 | 4 | 3 |

*Reps V, VI, VII, and VIII*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 3 | 4 | 9 | 1 | 8 | 6 | 5 | 7 |
| 5 | 6 | 7 | 2 | 9 | 1 | 4 | 3 | 8 |
| 7 | 5 | 9 | 1 | 6 | 3 | 8 | 4 | 2 |

**Plan 9B.8** $t = 9,\ k = 5, r = 10,\ b = 18,\ \lambda = 5,\ E = .90$

*Reps I, II, III, IV, and V*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 6 | 8 | 3 | 1 | 4 | 9 | 5 | 7 |
| 3 | 8 | 5 | 9 | 7 | 2 | 1 | 4 | 6 |
| 7 | 4 | 9 | 2 | 3 | 5 | 6 | 1 | 8 |
| 8 | 1 | 2 | 6 | 4 | 7 | 3 | 9 | 5 |

*Reps VI, VII, VIII, IX, and X*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 6 | 5 | 3 | 7 | 8 | 4 | 9 | 1 |
| 3 | 5 | 1 | 2 | 9 | 7 | 8 | 4 | 6 |
| 5 | 1 | 4 | 8 | 2 | 3 | 9 | 6 | 7 |
| 9 | 8 | 6 | 7 | 4 | 5 | 1 | 3 | 2 |

**Plan 9B.9** $t = 10,\ k = 3,\ r = 9,\ b = 30,\ \lambda = 2,\ E = .74$

*Reps I, II, and III*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 5 | 7 | 1 | 8 | 4 | 9 | 10 | 3 | 6 |
| 3 | 8 | 4 | 6 | 7 | 9 | 1 | 2 | 10 | 5 |

*Reps IV, V, and VI*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 3 | 4 | 9 | 7 | 8 | 10 | 1 | 5 | 6 |
| 4 | 6 | 8 | 5 | 1 | 9 | 3 | 10 | 2 | 7 |

*Reps VII, VIII, and IX*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | 7 | 8 | 2 | 6 | 1 | 9 | 4 | 10 | 5 |
| 5 | 6 | 9 | 10 | 3 | 8 | 2 | 7 | 1 | 4 |

**Plan 9B.10** $t = 11,\ k = 2, r = 10,\ b = 55,\ \lambda = 1,\ E = .55$

*Reps I and II*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 2 | 11 | 10 | 5 | 6 | 7 | 1 | 3 | 4 | 9 | 8 |

*Reps III and IV*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 6 | 5 | 10 | 9 | 8 | 2 | 1 | 7 | 11 | 4 |

*Reps V and VI*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 3 | 7 | 6 | 10 | 9 | 11 | 2 | 1 | 8 | 5 |

*Reps VII and VIII*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 5 | 9 | 6 | 2 | 7 | 10 | 8 | 4 | 11 | 1 | 3 |

*Reps IX and X*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 5 | 4 | 7 | 8 | 11 | 10 | 9 | 3 | 2 | 1 |



**Plan 9B.11** $t = 11,\ k = 5, r = 5,\ b = 11,\ \lambda = 2,\ E = .88$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | 9 | 11 | 10 | 8 | 2 | 6 | 3 | 5 | 4 |
| 2 | 1 | 8 | 9 | 11 | 7 | 6 | 3 | 4 | 10 | 5 |
| 3 | 6 | 1 | 7 | 5 | 2 | 4 | 11 | 10 | 9 | 8 |
| 4 | 10 | 6 | 1 | 8 | 3 | 11 | 5 | 9 | 2 | 7 |
| 5 | 3 | 2 | 4 | 1 | 11 | 10 | 9 | 8 | 7 | 6 |

**Plan 9B.12** $t = 11,\ k = 6,\ r = 6,\ b = 11,\ \lambda = 3,\ E = .92$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 5 | 4 | 3 | 2 | 1 | 9 | 8 | 7 | 11 | 10 |
| 7 | 8 | 5 | 10 | 3 | 6 | 1 | 2 | 11 | 4 | 9 |
| 8 | 4 | 7 | 2 | 9 | 10 | 3 | 1 | 5 | 6 | 11 |
| 9 | 11 | 3 | 6 | 7 | 4 | 5 | 10 | 1 | 8 | 2 |
| 10 | 2 | 11 | 5 | 4 | 9 | 8 | 7 | 6 | 1 | 3 |
| 11 | 9 | 10 | 8 | 6 | 5 | 7 | 4 | 2 | 3 | 1 |

**Plan 9B.13** $t = 13,\ k = 3,\ r = 6,\ b = 26,\ \lambda = 1,\ E = .72$

*Reps I, II, and III*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 | 2 |
| 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

*Reps IV, V, and VI*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 |

**Plan 9B.14** $t = 13,\ k = 4, r = 4,\ b = 13,\ \lambda = 1,\ E = .81$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 | 2 |
| 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

**Plan 9B.15** $t = 13,\ k = 9,\ r = 9,\ b = 13,\ \lambda = 6,\ E = .96$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 |
| 7 | 8 | 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 | 6 |
| 9 | 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 10 | 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 11 | 12 | 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 12 | 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 13 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

**Plan 9B.16** $t = 15,\ k = 7, r = 7,\ b = 15,\ \lambda = 3,\ E = .92$

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 13 | 5 | 15 | 12 | 4 | 11 | 1 | 2 | 8 | 10 | 9 | 14 | 7 | 3 | 6 |
| 8 | 14 | 12 | 11 | 5 | 9 | 2 | 3 | 6 | 4 | 13 | 7 | 15 | 1 | 10 |
| 12 | 10 | 11 | 6 | 8 | 7 | 3 | 1 | 4 | 5 | 14 | 13 | 9 | 2 | 15 |
| 6 | 7 | 5 | 9 | 1 | 2 | 4 | 13 | 15 | 11 | 10 | 3 | 12 | 8 | 14 |
| 7 | 12 | 8 | 2 | 14 | 13 | 5 | 15 | 10 | 1 | 6 | 4 | 3 | 9 | 11 |
| 1 | 2 | 3 | 4 | 9 | 15 | 6 | 14 | 13 | 12 | 5 | 8 | 10 | 11 | 7 |
| 9 | 8 | 6 | 14 | 15 | 5 | 7 | 12 | 2 | 13 | 3 | 11 | 4 | 10 | 1 |

**Plan 9B.17** $t = 15,\ k = 8,\ r = 8,\ b = 15,\ \lambda = 4,\ E = .94$

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 4 | 9 | 15 | 7 | 6 | 12 | 10 | 5 | 8 | 1 | 2 | 13 | 14 | 3 |
| 4 | 1 | 2 | 3 | 13 | 10 | 9 | 11 | 14 | 15 | 7 | 6 | 8 | 5 | 12 |
| 2 | 3 | 14 | 1 | 10 | 12 | 13 | 9 | 7 | 6 | 11 | 15 | 5 | 4 | 8 |
| 5 | 15 | 4 | 10 | 12 | 1 | 14 | 7 | 3 | 2 | 8 | 9 | 11 | 6 | 13 |
| 10 | 13 | 7 | 8 | 11 | 4 | 15 | 6 | 1 | 3 | 2 | 5 | 14 | 12 | 9 |
| 3 | 11 | 1 | 13 | 2 | 14 | 10 | 8 | 9 | 7 | 4 | 12 | 6 | 15 | 5 |
| 14 | 6 | 10 | 7 | 3 | 8 | 11 | 5 | 12 | 9 | 15 | 1 | 2 | 13 | 4 |
| 15 | 9 | 13 | 5 | 6 | 3 | 8 | 4 | 11 | 14 | 12 | 10 | 1 | 7 | 2 |

## 9A.3 Appendix: Least Squares Estimates for BIB Designs

The linear model for the balanced incomplete block design is

$$y_{ij} = \mu + \tau_i + \rho_j + e_{ij} \tag{9A.1}$$

$$i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, b$$

where $\mu$ is the general mean, $\tau_i$ is the treatment effect, $\rho_j$ is the block effect, and the $e_{ij}$ are random, independent experimental errors with mean 0 and variance $\sigma^2$.



Let $n_{ij}$ be an indicator variable with $n_{ij} = 1$ if the $i$th treatment is in the $j$th block and $n_{ij} = 0$ otherwise.

The least squares estimates for the parameters in the full model minimize the sum of squares for experimental error:

$$Q = \sum_{i=1}^{t}\sum_{j=1}^{b} n_{ij}e_{ij}^2 = \sum_{i=1}^{t}\sum_{j=1}^{b} n_{ij}(y_{ij} - \mu - \tau_i - \rho_j)^2 \tag{9A.2}$$

The normal equations for the balanced incomplete block design are

$$\mu:\quad N\widehat{\mu} + r\sum_{i=1}^{t}\widehat{\tau}_i + k\sum_{j=1}^{b}\widehat{\rho}_j = y_{..} \tag{9A.3}$$

$$\tau_i:\quad r\widehat{\mu} + r\widehat{\tau}_i + \sum_{j=1}^{b} n_{ij}\widehat{\rho}_j = y_{i.} \qquad i = 1, 2, \ldots, t \tag{9A.4}$$

$$\beta_j:\quad k\widehat{\mu} + \sum_{i=1}^{t} n_{ij}\widehat{\tau}_i + k\widehat{\rho}_j = y_{.j} \qquad j = 1, 2, \ldots, b \tag{9A.5}$$

The set of $t$ equations for the $\tau_i$ and the set of $b$ equations for the $\rho_j$ each sum to the equation for $\mu$, resulting in two linear dependencies among the equations. The constraints $\Sigma_i\widehat{\tau}_i = 0$ and $\Sigma_j\widehat{\rho}_j = 0$ can be used to provide a unique solution.

After the constraints are imposed, the estimate of $\mu$ is $\widehat{\mu} = y_{..}/N$ from Equation (9A.3). The equations for $\rho_j$, Equations (9A.5), are used to eliminate the $\widehat{\rho}_j$ from the equations for the $\tau_i$ in Equations (9A.4). After elimination the equations with $\widehat{\tau}_i$ are

$$rk\widehat{\tau}_i - r\widehat{\tau}_i - \sum_{j=1}^{b}\sum_{\substack{p=1\\p\neq i}}^{t} n_{ij}n_{pj}\widehat{\tau}_p = ky_{i.} - \sum_{j=1}^{b} n_{ij}y_{.j} \tag{9A.6}$$

The right-hand side of Equation (9A.6) is $kQ_i$, where $Q_i$ is the adjusted treatment total used to compute the adjusted treatment sum of squares. Since

$$\sum_{j=1}^{b} n_{ij}n_{pj} = \lambda, \text{ for } p \neq i \tag{9A.7}$$

and $n_{pj}^2 = n_{pj}$ ($n_{pj} = 0$ or 1), Equation (9A.6) can be written as

$$r(k-1)\widehat{\tau}_i - \lambda\sum_{\substack{p=1\\p\neq i}}^{t}\widehat{\tau}_p = kQ_i \tag{9A.8}$$

Since condition $\Sigma_i \widehat{\tau}_i = 0$ has been imposed on the solution, the substitution

$$-\widehat{\tau}_i = \sum_{\substack{p=1 \\ p \neq i}}^{t} \widehat{\tau}_p$$

can be made in Equation (9A.8). With the equality $\lambda(t-1) = r(k-1)$, the equation for $\widehat{\tau}_i$ is

$$\lambda t \widehat{\tau}_i = kQ_i \tag{9A.9}$$

$$\widehat{\tau}_i = \frac{kQ_i}{\lambda t} \quad i = 1, 2, \ldots, t$$

# 10 Incomplete Block Designs: Resolvable and Cyclic Designs

A general description of and analyses for incomplete block designs were introduced in Chapter 9. Several major classes of useful incomplete block designs illustrated in this chapter include resolvable designs with blocks grouped in complete replications of treatments. Cyclic designs that can be constructed without the use of extensive tabled plans and the $\alpha$ designs that extend the number of available resolvable designs for experiments are also discussed.

## 10.1 Resolvable Designs to Help Manage the Experiment

**Resolvable designs** have blocks grouped such that each group of blocks constitutes one complete replication of the treatments. The grouping into complete replications is useful in the management of an experiment.

One of the early applications for resolvable designs occurred with plant-breeding trials placed in field plots on experimental farmland. Plant breeders wanted to test a large number of genetic lines and to make all comparisons among pairs of lines with equal precision. Arrangement of field plots into blocks smaller than a complete replication was necessary to reduce experimental error variance more so than was possible with the complete block designs. The resolvable incomplete block design was attractive because it not only reduced the block sizes for greater precision, but it also allowed the researcher to manage these large studies in the field on a replication-by-replication basis.

The resolvable designs can be useful in practice when the entire experiment cannot be completed at one time. The experiment in a resolvable design can be conducted in stages, with one or more replications completed at each stage. In