5A Appendix: Coefficient Calculations for Expected Mean Squares in Table 5.9

$$A = \sum_{i=1}^{t}\sum_{j=1}^{r_i}\left(\frac{n_{ij}^2}{n_{i.}}\right)$$

$$= \frac{2^2 + \cdots + 1^2}{12} + \frac{1^2 + \cdots + 2^2}{13} + \frac{2^2 + \cdots + 2^2}{11} = 7.11655$$

$$B = \sum_{i=1}^{t}\sum_{j=1}^{r_i} n_{ij}^2 = 2^2 + 3^3 + \cdots + 1^2 + 2^2 = 86$$

$$D = \sum_{i=1}^{t} n_{i.}^2 = 12^2 + 13^2 + 11^2 = 434$$

$$c_1 = \frac{1}{t-1}\left(A - \frac{B}{N}\right) = \frac{1}{2}\left(7.11655 - \frac{86}{36}\right) = 2.36$$

$$c_2 = \frac{1}{t-1}\left(N - \frac{D}{N}\right) = \frac{1}{2}\left(36 - \frac{434}{36}\right) = 11.97$$

$$c_3 = \frac{1}{\left(\sum_{i=1}^{t} r_i - t\right)}(N - A) = \frac{1}{15}(36 - 7.11655) = 1.93$$

6 Factorial Treatment Designs

The factorial treatment design was introduced in Chapter 1 as a way to investigate the relationships among several types of treatments. The basic factorial treatment design in a completely randomized experiment design and its analysis are introduced in this chapter. Planned contrasts and response curve estimation, discussed in Chapter 3, are applied to the factorial treatment design. Methods to determine the number of required replications and to analyze the factorial treatment design with one replication or unequal treatment replications are discussed as well.

6.1 Efficient Experiments with Factorial Treatment Designs

Comparisons among treatments can be affected substantially by the conditions under which they occur. Frequently, clear interpretations of effects for one treatment factor must take into account the effects of other treatment factors. A special type of treatment design, **factorial treatment design**, was developed to investigate more than one factor at a time.

Factorial treatment designs produce efficient experiments. Each observation supplies information about all of the factors, and we are able to look at responses to one factor at different levels of another factor in the same experiment. The response to any factor observed under different conditions indicates whether the factors act on the experimental units independently of one another. Interaction between factors occurs when they do not act independently of one another.

Example 6.1 Compaction Effects on Asphaltic Concrete Durability

Asphalt pavements undergo water-associated deteriorations such as cracking, potholes, and surface raveling. The weakened pavement occurs when there is a break in the adhesive bond between aggregate and the asphaltic cements that make up the pavement. Research is directed to find improved asphalt pavements more resistant to deterioration.

The ability to develop superior asphalt pavement mixes requires a reliable method to test the experimental mix for bonding strength. Several methods have been developed to compact asphaltic pavement specimens in preparation for bonding strength tests.

Two factors known to have an effect on specimen bonding strength are (1) the methods used to compact the specimen during construction and (2) the aggregate type used in the asphalt mixture. If two compaction methods produce the same relative results for strength tests with two different aggregate types, then either compaction method could be used to evaluate experimental asphalt mixes for either aggregate type. If the results are dependent on aggregate type, then one or both of the compaction methods may not be adequate for discriminating between experimental mixes of asphalt.

The factorial treatment design can be used to evaluate whether the two factors act independently on the strength of the test specimens. The factorial arrangement is illustrated in Table 6.1 for test specimens prepared by two compaction methods (static and kneading) using two types of aggregate (silicious rock and basalt) for each compaction method. For illustration, specimen bonding tensile strength values are shown in Table 6.1 for the four treatments as pressure (psi) at test failure.

Factors are types of treatments such as compaction method and aggregate type, and different categories of a factor are levels of the factors. The levels of compaction method are Static and Kneading and the levels of aggregate type are Silicious Rock and Basalt. The factors are identified by uppercase letters A and B in Table 6.1. The levels of the factors are denoted $A_1, A_2, \ldots; B_1, B_2, \ldots$; and so forth. The factorial arrangement in Table 6.1 with two factors A and B each with two levels, has $2 \times 2 = 4$ treatment combinations, A_1B_1, A_1B_2, A_2B_1, and A_2B_2.

Table 6.1 Tensile strength (psi) of asphalt specimens

	Compaction Method (B)		
Aggregate Type (A)	*(B_1) Static*	*(B_2) Kneading*	*Aggregate Means*
Silicious (A_1)	68	60	64.0
Basalt (A_2)	65	97	81.0
Compaction Means	66.5	78.5	

Source: A. M. Al-Marshed (1981), Compaction effects on asphaltic concrete durability. M.S. thesis, Civil Engineering, University of Arizona.

The factorial treatment design consists of all possible combinations of the levels of several factors. Experiments with factorial treatment designs often are referred to as *factorials* or *factorial experiments*.

The levels of a *quantitative* factor take metrical values, whereas the levels of a *qualitative* factor are categories of the factor. Both factors in Example 6.1 are qualitative factors; the two levels of each factor are categories. Temperature exemplifies a quantitative factor with levels of 10°C, 20°C, and 30°C, for instance.

6.2 Three Types of Treatment Factor Effects

The **effect** of a factor is a change in the measured response caused by a change in the level of that factor. Three effects of interest in a factorial experiment are *simple effects, main effects,* and *interaction effects.* These effects are illustrated with population means for the factorial treatments in Example 6.1.

The population means for a factorial experiment with two factors, A and B, can be represented with cell means μ_{ij}. The term *cell mean* is derived from a tabled display of means for each of the treatment combinations, illustrated in Table 6.2 for a 2×2 factorial.

The means of the treatment combinations, μ_{11}, μ_{12}, μ_{21}, and μ_{22}, are located in the cells of the table—hence, the designation *cell means.* The means on the margins of the table are the averages of the cell means and are referred to as the *marginal means.* The overall or grand mean is the average of the cell means, $\overline{\mu}_{..} = \frac{1}{4}(\mu_{11} + \mu_{12} + \mu_{21} + \mu_{22})$.

Table 6.2 Table of means for a 2×2 factorial experiment

	B		
A	1	2	Factor A Means
1	μ_{11}	μ_{12}	$\overline{\mu}_{1.} = \frac{1}{2}(\mu_{11} + \mu_{12})$
2	μ_{21}	μ_{22}	$\overline{\mu}_{2.} = \frac{1}{2}(\mu_{21} + \mu_{22})$
Factor B Means	$\overline{\mu}_{.1} = \frac{1}{2}(\mu_{11} + \mu_{21})$	$\overline{\mu}_{.2} = \frac{1}{2}(\mu_{12} + \mu_{22})$	

Simple Effects Are Contrasts

The **simple effects** of a factor are contrasts between levels of one factor at a single level of another factor. The simple effect (l_1) of aggregate type (A) on tensile strength with static compaction (B_1) calculated from the cell means in Table 6.1 is

$$l_1 = \mu_{21} - \mu_{11} = 65 - 68 = -3$$

It measures the difference in tensile strength between basalt and silicious rock specimens when the static compaction method is used to form the specimens. The average tensile strength of silicious rock specimens was greater than that for basalt specimens. Similarly, the simple effect

$$l_2 = \mu_{22} - \mu_{12} = 97 - 60 = 37$$

measures the difference in tensile strength between basalt and silicious rock specimens when the kneading compaction method is used. In this case, the average tensile strength of the basalt specimens was greater than that for the silicious rock specimens.

Main Effects Are Average Effects of a Factor

The **main effects** of a factor are contrasts between levels of one factor averaged over all levels of another factor. The main effect of aggregate type on tensile strength is the difference between the marginal means for aggregate type in Table 6.2:

$$l_3 = \overline{\mu}_{2.} - \overline{\mu}_{1.} = 81 - 64 = 17 \qquad \textbf{(6.1)}$$

The difference in tensile strength between basalt and silicious rock specimens is 17 psi in favor of the basalt when averaged over both compaction methods.

Upon close inspection the main effect for aggregate type can be expressed as the average of the two simple effects. Thus, from Equation (6.1)

$$\begin{aligned} l_3 = \overline{\mu}_{2.} - \overline{\mu}_{1.} &= \frac{1}{2}(\mu_{21} + \mu_{22}) - \frac{1}{2}(\mu_{11} + \mu_{12}) \\ &= \frac{1}{2}(\mu_{22} - \mu_{12}) + \frac{1}{2}(\mu_{21} - \mu_{11}) \\ &= \frac{1}{2}(l_1 + l_2) \end{aligned}$$

or $17 = \frac{1}{2}(-3 + 37)$. The main effect contrast, $l_3 = 17$, implies the basalt specimens are stronger than the silicious rock specimens. The simple effect contrast for kneading compaction, $l_2 = 37$, supports the same conclusion. However, the simple effect contrast for static compaction, $l_1 = -3$, suggests the opposite conclusion. The difference between the two simple effects indicates aggregate type and compaction method do not act independently of one another in their influence on specimen strength.

Interaction Effects Are Differences Between Simple Effects

The **interaction effect** measures differences between the simple effects of one factor at different levels of the other factor. Consider the two simple effects of aggregate type on tensile strength shown in Display 6.1.

The difference between the two simple effects, $l_4 = l_2 - l_1$, measures interaction between aggregate type and compaction method factors as they affect the tensile strength of the specimens. The difference between basalt and silicious rock specimens was 40 units greater with kneading compaction than it was with static

Display 6.1 Interaction Effect of Compaction Method with Aggregate Type

Compaction Method	*Simple Effect of Aggregate Type*
Static	$l_1 = 65 - 68 = -3$
Kneading	$l_2 = 97 - 60 = 37$
Difference	$l_4 = 37 - (-3) = 40$

compaction. Thus, the effect of aggregate type on tensile strength measurements of asphalt is dependent on the method employed to compact the specimen for testing.

The example illustrates the caution that must be exercised in interpreting main effects and represents a situation in which interpretations should not be based on main effects. The effect of aggregate type on tensile strength of the specimens differed considerably between the static and kneading compaction methods. Although the main effect measurement suggested the basalt specimens would be stronger, it was only true for the kneading compaction method.

The 2×2 factorial interaction contrast is derived from the difference between the two simple effects for factor A.

$$\begin{aligned} AB &= (\mu_{22} - \mu_{12}) - (\mu_{21} - \mu_{11}) \\ &= (\mu_{22} - \mu_{21} - \mu_{12} + \mu_{11}) \end{aligned} \qquad \textbf{(6.2)}$$

The same expression may be derived from the simple effects for B.

The presence or absence of interaction is illustrated graphically in Figures 6.1 and 6.2 for a factorial arrangement with two factors, A and B, each at two levels. The graphic measure of a simple effect for each factor is illustrated in Figure 6.1. The response to A is graphed separately for each level of B. In the absence of interaction the parallel lines show an identical response to A for both levels of B. Under these circumstances, the factors act independently, and the main effects can be used to interpret the effects of the two factors separately.

The presence of interaction is illustrated in Figure 6.2. The response lines are not parallel when there is an interaction between the two factors. Differences in the magnitude of the responses (Figure 6.2a) or in direction of the responses (Figure 6.2b) represent interaction between the factors. The factors do not act independently, and interpretations should be based on simple effect contrasts.

Example 6.2 Compaction Effects on Asphaltic Concrete Revisited

Research Objectives: The variation in tensile strength of asphaltic concrete test specimens, as discussed in Example 6.1, was known to be associated with the compaction method and aggregate type used to construct the specimens. A civil engineer conducted an experiment to evaluate differences among a set of compaction methods for their effect on the tensile strength of test specimens and to determine to what extent the aggregate type affected the comparisons among the compaction methods.

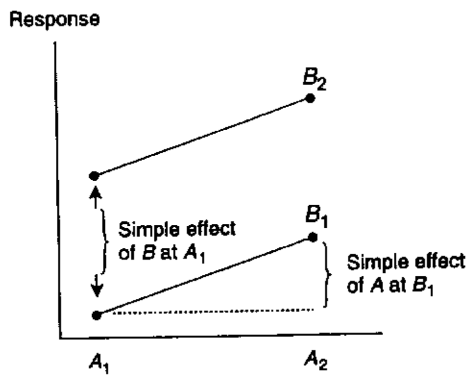

Figure 6.1 Illustration of no interaction in a factorial arrangement

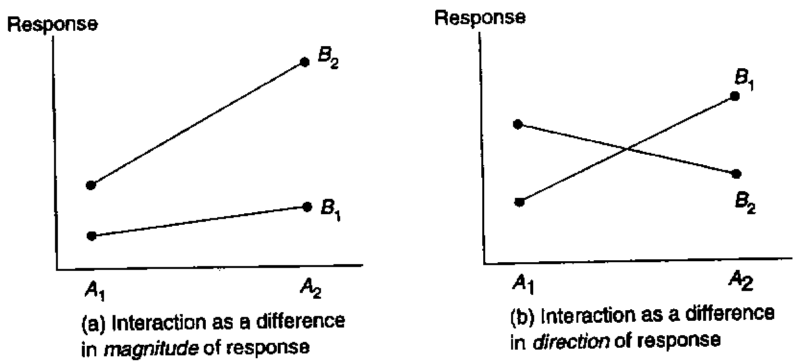

Figure 6.2 Illustration of interaction in a factorial arrangement

Treatment Design: A factorial arrangement was used with "compaction method" and "aggregate type" as factors. There were two levels of aggregate type—A_1 (basalt) and A_2 (silicious rock)—and four levels of compaction method—C_1 (static pressure), C_2 (regular kneading), C_3 (low kneading), and C_4 (very low kneading).

Experiment Design: Three replicate specimens of the asphalt concrete were constructed and tested for each of the eight treatment combinations. The 24 specimens were prepared and tested in random order for a completely randomized design.

The data for tensile strength measurements (psi) on the 24 specimens are shown in Table 6.3.

Table 6.3 Tensile strength (psi) of asphaltic concrete specimens for two aggregate types with each of four compaction methods

	Compaction Method			
		Kneading		
Aggregate Type	*Static*	*Regular*	*Low*	*Very Low*
Basalt	68	126	93	56
	63	128	101	59
	65	133	98	57
Silicious	71	107	63	40
	66	110	60	41
	66	116	59	44

Source: A. M. Al-Marshed (1981), Compaction effects on asphaltic concrete durability. M.S. thesis, Civil Engineering, University of Arizona.

6.3 The Statistical Model for Two Treatment Factors

The Cell Means Model

The observations from a factorial experiment with two factors, A with a levels and B with b levels, can be represented with the **cell means model**. The cell means model for the $a \times b$ factorial with r replications in a completely randomized design is

$$y_{ijk} = \mu_{ij} + e_{ijk} \tag{6.3}$$

$$i = 1, 2, \ldots, a \quad j = 1, 2, \ldots, b \quad k = 1, 2, \ldots, r$$

where μ_{ij} is the mean of the treatment combination A_iB_j and e_{ijk} are random experimental errors with mean 0 and variance σ^2.

Least Squares Estimates of Cell Means

The cell means can be estimated by the least squares method outlined in Chapter 2. The sum of squares for experimental error is

$$SS\text{ Error} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}\widehat{e}_{ijk}^{2} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r}(y_{ijk} - \widehat{\mu}_{ij})^2 \tag{6.4}$$

The least squares estimators for μ_{ij} are the observed cell means of the treatment combinations.

$$\widehat{\mu}_{ij} = \frac{y_{ij.}}{r} = \overline{y}_{ij.} \tag{6.5}$$

$$i = 1, 2, \ldots, a \qquad j = 1, 2, \ldots, b$$

The observed marginal means are unbiased estimates of the factor marginal means, so that $\widehat{\overline{\mu}}_{i.} = \overline{y}_{i..}$ and $\widehat{\overline{\mu}}_{.j} = \overline{y}_{.j.}$. The overall mean $\overline{\mu}_{..}$ is estimated with the observed grand mean, $\overline{y}_{...}$. Estimated cell and marginal means for the asphaltic concrete specimens are shown in Table 6.4.

Table 6.4 Estimated cell and marginal means for tensile strength of asphaltic concrete specimens

	Compaction Method				
		Kneading			*Aggregate*
Aggregate Type	*Static*	*Regular*	*Low*	*Very Low*	*Means* ($\overline{y}_{i..}$)
Basalt	65.3	129.0	97.3	57.3	87.3
Silicious	67.7	111.0	60.7	41.7	70.3
Compaction means ($\overline{y}_{.j.}$)	66.5	120.0	79.0	49.5	$\overline{y}_{...} = 78.8$

Additivity and Factor Effects

The cell means μ_{ij} represent the true response for the treatment combination of level i for A and level j for B. In the absence of interaction the cell mean can be expressed as a sum of a general mean, μ, plus an effect contributed by A, say α_i, and an effect contributed by B, say β_j, so that $\mu_{ij} = \mu + \alpha_i + \beta_j$.

The effect for the ith level of factor A can be defined as $\alpha_i = \overline{\mu}_{i.} - \overline{\mu}_{..}$. The effect of A is a deviation of the marginal mean from the grand mean. The effect for the jth level of B can be defined as $\beta_j = \overline{\mu}_{.j} - \overline{\mu}_{..}$. The effects will be **fixed effects** if the levels of the factors are reproducible. In the absence of interaction, the cell mean is the sum of the grand mean and the effects of the factors for that cell:

$$\mu_{ij} = \overline{\mu}_{..} + (\overline{\mu}_{i.} - \overline{\mu}_{..}) + (\overline{\mu}_{.j} - \overline{\mu}_{..}) \tag{6.6}$$

The effect of the ijth treatment combination, $\tau_{ij} = (\mu_{ij} - \overline{\mu}_{..})$, is the sum of the two factor effects:

$$(\mu_{ij} - \overline{\mu}_{..}) = (\overline{\mu}_{i.} - \overline{\mu}_{..}) + (\overline{\mu}_{.j} - \overline{\mu}_{..}) \tag{6.7}$$

and the effects of the factors are *additive* in the absence of interaction.

In the presence of interaction the treatment effect will not be equal to the sum of the individual factor effects as shown in Equation (6.7). An interaction effect, denoted as $(\alpha\beta)_{ij}$, can be defined as the difference between the two sides of Equation (6.7), or

$$\begin{aligned}(\alpha\beta)_{ij} &= (\mu_{ij} - \overline{\mu}_{..}) - (\overline{\mu}_{i.} - \overline{\mu}_{..}) - (\overline{\mu}_{.j} - \overline{\mu}_{..}) \\ &= \mu_{ij} - \overline{\mu}_{i.} - \overline{\mu}_{.j} + \overline{\mu}_{..}\end{aligned} \tag{6.8}$$

The new set of parameters, α_i, β_j, and $(\alpha\beta)_{ij}$, can be used to write the linear model for the factorial as an *effects* model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \tag{6.9}$$

$$i = 1, 2, \ldots, a \qquad j = 1, 2, \ldots, b \qquad k = 1, 2, \ldots, r$$

where μ is the grand mean $\overline{\mu}_{..}$, α_i is the effect of the ith level of A, β_j is the effect of the jth level of B, and $(\alpha\beta)_{ij}$ is the interaction effect between the ith level of A and the jth level of B. By the nature of their definitions the sums of the effects are equal to zero. That is,

$$\sum_{i=1}^{a} \alpha_i = 0, \quad \sum_{j=1}^{b} \beta_j = 0, \quad \sum_{i=1}^{a} (\alpha\beta)_{ij} = \sum_{j=1}^{b} (\alpha\beta)_{ij} = 0 \tag{6.10}$$

6.4 The Analysis for Two Factors

Fundamental Sum of Squares Partition

The fundamental partition of the total sum of squares can be derived from the equation

$$(y_{ijk} - \overline{y}_{...}) = (\overline{y}_{ij.} - \overline{y}_{...}) + (y_{ijk} - \overline{y}_{ij.}) \tag{6.11}$$

The deviation of an observation from the grand mean $(y_{ijk} - \overline{y}_{...})$ is the sum of two parts:

- the treatment effect $(\overline{y}_{ij.} - \overline{y}_{...})$
- experimental error $(y_{ijk} - \overline{y}_{ij.})$

For example, using the observations in Table 6.3 and the means in Table 6.4, the total deviation for the first observation in Table 6.3 is

$$(y_{111} - \overline{y}_{...}) = 68 - 78.8 = -10.8.$$

The treatment deviation is

$$(\overline{y}_{11.} - \overline{y}_{...}) = (65.3 - 78.8) = -13.5$$

and the experimental error is

$$(y_{111} - \overline{y}_{11.}) = (68 - 65.3) = 2.7$$

and

$$-10.8 = -13.5 + 2.7$$

The latter deviation $(y_{ijk} - \bar{y}_{ij.})$ is a measure of experimental error for the observation in a properly replicated experiment. If both sides of Equation (6.11) are squared and summed over all observations, the result is

$$\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{r} (y_{ijk} - \bar{y}_{...})^2 = r \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{ij.} - \bar{y}_{...})^2 + \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{r} (y_{ijk} - \bar{y}_{ij.})^2 \quad \textbf{(6.12)}$$

or

$$SS \text{ Total} = SS \text{ Treatment} + SS \text{ Error}$$

Any crossproducts formed by squaring the right-hand side of Equation (6.11) sum to zero. There are a total of rab observations, so that SS Total has $(rab - 1)$ degrees of freedom. With $t = ab$, SS Treatment has $(ab - 1)$ degrees of freedom and the remaining $ab(r - 1)$ degrees of freedom are associated with SS Error.

Sums of Squares for Factorial Effects

The treatment effect $(\bar{y}_{ij.} - \bar{y}_{...})$ in Equation (6.11) can be expressed as the identity

$$(\bar{y}_{ij.} - \bar{y}_{...}) = (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) \quad \textbf{(6.13)}$$

where the treatment effect is a sum of three effects:

- factor A main effect $(\bar{y}_{i..} - \bar{y}_{...})$
- factor B main effect $(\bar{y}_{.j.} - \bar{y}_{...})$
- interaction $(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})$

For example, the treatment effect for basalt with low kneading in Table 6.4 is

$$(\bar{y}_{13.} - \bar{y}_{...}) = 97.3 - 78.8 = 18.5$$

The main and interaction effects are

- Basalt main effect :

$$(\bar{y}_{1..} - \bar{y}_{...}) = 87.3 - 78.8 = 8.5$$

- Low kneading main effect:

$$(\bar{y}_{.3.} - \bar{y}_{...}) = 79.0 - 78.8 = 0.2$$

- Interaction:

$$(\bar{y}_{13.} - \bar{y}_{1..} - \bar{y}_{.3.} + \bar{y}_{...}) = 97.3 - 87.3 - 79.0 + 78.8 = 9.8$$

and

$$18.5 = 8.5 + 0.2 + 9.8$$

If both sides of Equation (6.13) are squared and summed over all observations the left-hand side is the SS Treatment. The treatment sum of squares is partitioned into three components represented by the effects on the right-hand side of Equation (6.13). Any crossproducts formed by squaring the right-hand side will sum to zero.

The sum of squares for the first component will be the sum of squares among the marginal means for A

$$SSA = rb \sum_{i=1}^{a} (\bar{y}_{i..} - \bar{y}_{...})^2 \quad \textbf{(6.14)}$$

and the second will be the sum of squares among the marginal means for B

$$SSB = ra \sum_{j=1}^{b} (\bar{y}_{.j.} - \bar{y}_{...})^2 \quad \textbf{(6.15)}$$

The sum of squares for the third component

$$SS(AB) = r \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 \quad \textbf{(6.16)}$$

is the sum of squares for interaction. That part of the treatment sum of squares is not explained by the sum of squares attributed to the two factors as SSA and SSB. Consequently, the additive partition of SS Treatment is

$$SS \text{ Treatment} = SSA + SSB + SS(AB)$$

The $(ab - 1)$ degrees of freedom for the treatment sum of squares are allocated to the three partitions of SS Treatment. The factors, A and B, have a and b levels respectively, therefore SSA and SSB have $(a - 1)$ and $(b - 1)$ degrees of freedom. The remaining degrees of freedom allocated to the sum of squares for interaction are the degrees of freedom for treatments $(ab - 1)$ minus the degrees of freedom for the separate factor sums of squares $(a - 1)$ and $(b - 1)$, or $(ab - 1) - (a - 1) - (b - 1) = (a - 1)(b - 1)$. The degrees of freedom for interaction sums of squares in factorials is the product of the degrees of freedom for the factors included in the interaction.

The complete partition of the total sum of squares for a factorial arrangement with two factors is summarized in the analysis of variance shown in Table 6.5.

The derivation of the sum of squares partitions from solutions for the least squares normal equations of the effects model is shown in Appendix 6A. The

Table 6.5 Analysis of variance for a two-factor treatment design

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*Expected Mean Square*
Total	$rab-1$	SS Total		
Factor A	$a-1$	SSA	MSA	$\sigma_e^2 + rb\theta_a^2$
Factor B	$b-1$	SSB	MSB	$\sigma_e^2 + ra\theta_b^2$
AB Interaction	$(a-1)(b-1)$	$SS(AB)$	$MS(AB)$	$\sigma_e^2 + r\theta_{ab}^2$
Error	$ab(r-1)$	SSE	MSE	σ_e^2

$$\theta_a^2 = \sum_{i=1}^{a} (\overline{\mu}_{i.} - \overline{\mu}_{..})^2/(a-1) \qquad \theta_b^2 = \sum_{j=1}^{b} (\overline{\mu}_{.j} - \overline{\mu}_{..})^2/(b-1)$$

$$\theta_{ab}^2 = \sum_{i=1}^{a}\sum_{j=1}^{b} (\mu_{ij} - \overline{\mu}_{i.} - \overline{\mu}_{.j} + \overline{\mu}_{..})^2/(a-1)(b-1)$$

Table 6.6 Analysis of variance for tensile strength of asphalt specimens in a 4×2 factorial arrangement

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	F	$Pr > F$
Total	23	19,274.50			
Compaction (C)	3	16,243.50	5,414.50	569.95	.000
Aggregate (A)	1	1,734.00	1,734.00	182.53	.000
Interaction (AC)	3	1,145.00	381.67	40.18	.000
Error	16	152.00	9.50		

analysis of variance for the data from Table 6.3 on the asphalt concrete specimens is shown in Table 6.6.

Tests of Hypotheses About Factor Effects

Inferences about individual factor effects depend upon the presence or absence of interaction. Significance of interaction is determined before any determinations of significance for main effects of the factors.

In the absence of interaction, $(\alpha\beta)_{ij} = \mu_{ij} - \overline{\mu}_{i.} - \overline{\mu}_{.j} + \overline{\mu}_{..} = 0$ from Equation (6.8), and $\theta_{ab}^2 = 0$ in the expected mean square for AB interaction. The null hypothesis for interaction

$$H_0\colon (\alpha\beta)_{ij} = \mu_{ij} - \overline{\mu}_{i.} - \overline{\mu}_{.j} + \overline{\mu}_{..} = 0 \quad \text{for all } i, j$$

versus the alternative

$$H_a\colon (\alpha\beta)_{ij} = \mu_{ij} - \overline{\mu}_{i.} - \mu_{.j} + \overline{\mu}_{..} \neq 0 \quad \text{for some } i, j$$

is tested with

$$F_0 = \frac{MS(AB)}{MSE} \tag{6.17}$$

with critical value $F_{\alpha,(a-1)(b-1),ab(r-1)}$.

If there are no differences among the marginal means for A, then $\alpha_i = \overline{\mu}_{i.} - \overline{\mu}_{..} = 0$ and $\theta_a^2 = 0$ in the expected mean square for A. The null hypothesis of equality among the means

$$H_0\colon \overline{\mu}_{1.} = \overline{\mu}_{2.} = \cdots = \overline{\mu}_{a.}$$

versus the alternative

$$H_a\colon \overline{\mu}_{i.} \neq \overline{\mu}_{k.} \quad \text{for some } i, k$$

is tested with the ratio

$$F_0 = \frac{MSA}{MSE} \tag{6.18}$$

with critical value $F_{\alpha,(a-1),ab(r-1)}$.

If there are no differences among the marginal means for B, then $\beta_j = \overline{\mu}_{.j} - \overline{\mu}_{..} = 0$ and $\theta_b^2 = 0$ in the expected mean square for B. The null hypothesis of equality among the means

$$H_0\colon \overline{\mu}_{.1} = \overline{\mu}_{.2} = \cdots = \overline{\mu}_{.b}$$

versus the alternative

$$H_a\colon \overline{\mu}_{.j} \neq \overline{\mu}_{.m} \quad \text{for some } j, m$$

is tested with the ratio

$$F_0 = \frac{MSB}{MSE} \tag{6.19}$$

with critical value $F_{\alpha,(b-1),ab(r-1)}$.

F Tests for Aggregate Type and Compaction Method Effects

The F test for interaction, $F_0 = MS(AC)/MSE = 381.67/9.50 = 40.18$ in Table 6.6, indicates a significant interaction between aggregate type and compaction method since F_0 exceeds $F_{.05,3,16} = 3.24$. The marginal means for aggregate type are significantly different since $F_0 = MSA/MSE = 1734.00/9.50 = 182.53$ exceeds $F_{.05,1,16} = 4.49$. The marginal means for compaction method are also different since $F_0 = MSC/MSE = 5414.50/9.50 = 569.95$ exceeds $F_{.05,3,16} = 3.24$. The significance level for each of the tests is listed as $Pr > F = .000$ in the rightmost column of Table 6.6.

The significant interaction can modify any inferences based on the significant differences among the marginal means of aggregate and compaction. The summary table of cell and marginal means in Table 6.4 will aid in the interpretation of the results.

Standard Errors and Interval Estimates for Means

The standard errors for the estimated marginal and cell means of the factorial experiment are

$$\text{Aggregate:}\quad s_{\overline{y}_{i..}} = \sqrt{\frac{MSE}{rb}} = \sqrt{\frac{9.5}{(3)(4)}} = 0.89$$

$$\text{Compaction:}\quad s_{\overline{y}_{.j}} = \sqrt{\frac{MSE}{ra}} = \sqrt{\frac{9.5}{(3)(2)}} = 1.26 \qquad \textbf{(6.20)}$$

$$\text{Cell means:}\quad s_{\overline{y}_{ij.}} = \sqrt{\frac{MSE}{r}} = \sqrt{\frac{9.5}{3}} = 1.78$$

The Student t with $ab(r-1)$ degrees of freedom is required for interval estimates of the marginal and cell means. A 95% confidence interval estimate requires $t_{.025,16} = 2.12$. The interval estimate for a cell mean is

$$\overline{y}_{ij.} \pm t_{.025,16}(s_{\overline{y}_{ij.}}) \qquad \textbf{(6.21)}$$

For example, the 95% confidence interval estimate for the basalt aggregate with static compaction mean is

$$65.3 \pm 2.12(1.78)$$
$$(61.5,\ 69.1)$$

Interval estimates are calculated similarly for the other means upon substitution of the appropriate mean and standard error estimate from Equation (6.20).

Multiple Contrasts Assist Interpretations of Significant Interaction Effects

The significant interaction between aggregate type and compaction method indicates the simple effects of one factor differ among levels of another factor. Consequently, tests of hypotheses about the treatment factors initially should be based on simple effect contrasts among the cell means.

The specific research hypotheses for the study will dictate the contrasts among the cell means required to investigate the simple effects. One general research question for this study might ask which of the aggregate types results in the strongest specimens for each of the compaction methods. Another hypothesis might address the effect of the kneading compaction methods relative to static compaction.

Contrasts among the cell means in Table 6.4 can be used to test the aggregate type simple effects (basalt versus silicious) at each level of compaction to address the question of which aggregate type results in the strongest specimens for each

compaction method. The four contrasts and their 95% simultaneous confidence intervals are shown in Display 6.2. The experimentwise error rate for the family of four tests can be controlled with the Bonferroni t statistic, $t_{.025,4,16} = 2.81$ from Appendix Table V.

Display 6.2 Estimated Contrasts for Aggregate Simple Effects

Compaction Method	*Contrast (Basalt–Silicious)*	*95% SCI (L, U)*
Static	$c_1 = \overline{y}_{11.} - \overline{y}_{21.} = 65.3 - 67.7 = -2.4$	$(-9.5, 4.7)$
Regular	$c_2 = \overline{y}_{12.} - \overline{y}_{22.} = 129.0 - 111.0 = 18.0$	$(10.9, 25.1)$
Low	$c_3 = \overline{y}_{13.} - \overline{y}_{23.} = 97.3 - 60.7 = 36.6$	$(29.5, 43.7)$
Very low	$c_4 = \overline{y}_{14.} - \overline{y}_{24.} = 57.3 - 41.7 = 15.6$	$(8.5, 22.7)$

Standard error $s_c = \sqrt{\frac{MSE}{r}[1^2 + (-1)^2]} = \sqrt{\frac{9.5}{3}(2)} = 2.52$

Summary Statements About Effects of the Factors

There is no difference in specimen tensile strength between the two aggregate types with static compaction since the interval includes 0. The specimens constructed from basalt have greater tensile strength than those constructed from the silicious rock for the kneading compaction methods since lower limits of the 3 intervals are greater than 0 and the greatest difference between the aggregate types was found with the low kneading compaction.

The comparisons to consider depend on the nature of the problem and the information required from the study. Those comparisons discussed in the preceding paragraphs were made to illustrate the interpretation of interaction. They exploited the notion that inferences regarding comparisons between aggregate types or among compaction methods depend on the other factor in the study.

A comparison among the marginal means for a factor can be informative when all of its simple effects are of a similar direction and magnitude. The comparisons among marginal means result in a more general inference about the factor, and they are more precise with smaller standard errors; see Equation (6.20). In the presence of interaction caution must be exercised in making generalizations. The difference between the marginal means for basalt and silicious aggregate types, $c = (87.3 - 70.3) = 17.0$, with standard error $s_c = \sqrt{2MSE/rb} = \sqrt{2(9.5)/12} = 1.26$, will be significant. The main effect estimate is a difference averaged over all compaction methods. The simple effects shown in Display 6.2 for aggregate type range from $c_1 = -2.4$ to $c_3 = 36.6$, depending on the compaction method. Generalizations about aggregate types based on the main effect estimate would be misleading.

Residual Analysis to Evaluate Assumptions

The assumptions for the model regarding homogeneity of variances and normal distribution can be evaluated with the residuals as discussed in Chapter 4. The residuals for the two-factor factorial are computed as the deviations of the observed values from the estimated means for each cell in the arrangement. The residual for any cell is $\hat{e}_{ijk} = y_{ijk} - \hat{y}_{ijk} = y_{ijk} - \bar{y}_{ij.}$ For example, the residual for the first observation in Table 6.3 is

$$\hat{e}_{111} = y_{111} - \bar{y}_{11.} = 68 - 65.3 = 2.7$$

The plots for the square root of absolute residuals versus the estimated values and the normal probability plot of the residuals are shown in Figures 6.3a and b. The plots show no strong evidence of heterogeneous variance or nonnormality. The Levene (Med) test for homogeneity of variance is left as an exercise for the reader with reference to Chapter 4 on methods for evaluating assumptions.

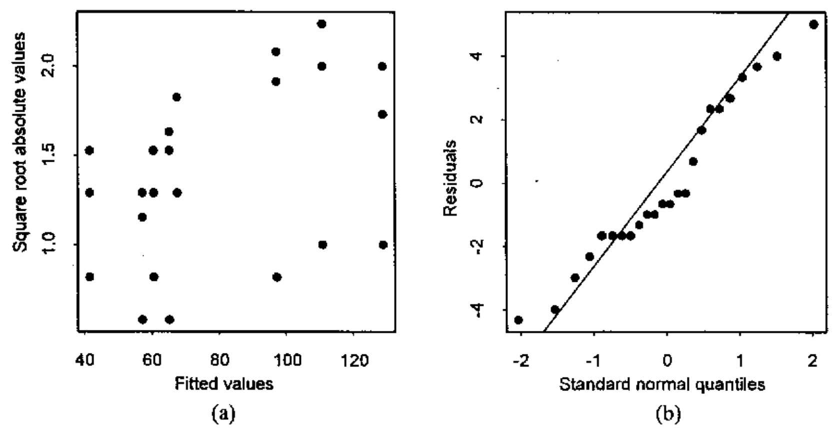

Figure 6.3 Residual plots from the analysis of variance for data on strength in Example 6.2: (a) square root of the absolute residuals vs. estimated values and (b) normal probability plot of residuals

6.5 Using Response Curves for Quantitative Treatment Factors

Response trend curves for quantitative treatment factors were estimated with orthogonal polynomials in Section 3.3. Recall that the estimated response curve has the advantage of portraying the relationship between the response variable and the treatment factor throughout the numerical range of the factor that was used in the study. The evaluation of response trend curves is extended to the two-factor experiment in this section. The analysis is discussed initially for experiments with

one quantitative factor and one qualitative factor. Subsequently, responses will be analyzed for experiments with two quantitative factors.

One Quantitative Factor and One Qualitative Factor

Ascertaining whether there is interaction between the quantitative and qualitative factor is the first objective in the analysis. The response to the quantitative factor will be different across levels of the qualitative factor when the two factors interact. In that case, the response curves for the quantitative factor can be estimated separately for each level of the qualitative factor. In the absence of interaction, the response trend to the quantitative factor will be similar at all levels of the qualitative factor, and a single response curve will suffice for description of the process with respect to the quantitative factor.

Example 6.3 Heavy Metals in Sewage Sludge

Sludge is a dried product remaining from processed sewage; it contains nutrients beneficial to plant growth. It can be used for fertilizer on agricultural crops provided it does not contain toxic levels of certain elements such as heavy metals. Typically, the levels of metals in sludge are assayed by growing plants in media containing different doses of the sludge.

Research Hypothesis: A soil scientist hypothesized the concentration of certain heavy metals in sludge would differ among the metropolitan areas from which the sludge was obtained. The variation could be a result of any number of causes, including different industrial bases surrounding the areas. If this were true, then recommendations for applications on crops would have to be preceded by knowledge about the source of the material. An assay was planned to determine whether there was significant variation in heavy metal concentrations among diverse metropolitan areas.

Treatment Design: The investigator obtained sewage sludge from treatment plants located in three different metropolitan areas. Barley plants were grown in a sand medium to which the sludge was added as a fertilizer. The sludge was added to the sand at three different rates: 0.5, 1.0, and 1.5 metric tons/acre. The factorial arrangement for the treatment design consisted of one qualitative factor, "city," with three levels and one quantitative factor, "rate," with three levels.

Experiment Design: Each of the nine treatment combinations was assigned to four replicate containers in a completely randomized design. The containers were arranged completely at random in a growth chamber. At a certain stage of growth the zinc content in parts per million was determined for the barley plants grown in each of the containers. The data are shown in Table 6.7, and the analysis of variance is shown in Table 6.8. The manual calculations for linear and quadratic sums of squares partitions are shown in Table 6.9.

Table 6.7 Zinc content (ppm) of barley plants grown in media containing sludge at three rates from three metropolitan areas

City and Rate(MT/hectare)								
A			B			C		
0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
26.4	25.2	26.0	30.1	47.7	73.8	19.4	23.2	18.9
23.5	39.2	44.6	31.0	39.1	71.1	19.3	21.3	19.8
25.4	25.5	35.5	30.8	55.3	68.4	18.7	23.2	19.6
22.9	31.9	38.6	32.8	50.7	77.1	19.0	19.9	21.9

Source: J. Budzynski, Department of Soil and Water Science, University of Arizona.

Table 6.8 Analysis of variance for zinc content in barley plants grown in media containing sewage sludge at three different rates from three metropolitan areas

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	35	9993.38			
Rate (R)	2	1945.45	972.72	50.71	.000
Rate linear	1	1944.00	1944.00	101.35	.000
Rate quadratic	1	1.45	1.45	0.08	.786
City (C)	2	5720.67	2860.34	149.13	.000
Rate × City (RC)	4	1809.40	452.35	23.58	.000
Rate linear × City (RC)	2	1760.15	880.07	45.88	.000
Rate quadratic × City	2	49.25	24.63	1.28	.293
Error	27	517.86	19.18		

The analysis of variance in Table 6.8 indicates significant interaction between Rate and City and significant main effects for both factors ($Pr > F = .000$). The 2 degrees of freedom for Rate sum of squares partition into 1 degree of freedom each for the linear and quadratic rates. The F test indicates significance for the linear regression partition ($F_0 = 101.35$) and nonsignificance ($F_0 = 0.08$) for the quadratic partition for Rate.

The 2 degrees of freedom for each of the Rate by City interaction sums of squares indicate the variability among cities in linear and quadratic regression coefficients for Rate. The interaction between Rate linear regression and City is significant ($F_0 = 45.88$), but the interaction between Rate quadratic regression and City is not significant ($F_0 = 1.28$).

Interpret Factor Effects with Regression Contrasts

The significant interactions between city and linear regression on rate of sludge application suggests that interpretations should be based on separate regression lines for each city. The estimated linear regression lines for each city are plotted in

Table 6.9 Calculation of linear and quadratic contrast sums of squares partitions for rate and rate × city interaction

City	Rate (metric tons/hectare) 0.5	1.0	1.5	Linear $\Sigma P_{1j}\overline{y}_{ij.}$	Quadratic $\Sigma P_{2j}\overline{y}_{ij.}$
A	24.55	30.45	36.18	11.63	−0.17
B	31.18	48.20	72.60	41.42	7.38
C	19.10	21.90	20.05	0.95	−4.65
Means ($\overline{y}_{.j.}$)	24.94	33.52	42.94	18.00	0.84
Linear (P_{1j})	−1	0	1		
Quadratic (P_{2j})	1	−2	1		

$$SS[R \text{ linear}] = ra[\Sigma P_{1j}\overline{y}_{.j.}]^2/\Sigma P_{1j}^2 = 12[18]^2/2 = 1944$$

$$SS[R \text{ quadratic}] = ra[\Sigma P_{2j}\overline{y}_{.j.}]^2/\Sigma P_{2j}^2 = 12[0.84]^2/6 = 1.41$$

$$SS[R \text{ linear} \times C] = r\sum_i[\sum_j P_{1j}\overline{y}_{ij.}]^2/\Sigma P_{1j}^2 - SS[R \text{ linear}]$$

$$= \frac{4[11.63^2 + 41.42^2 + 0.95^2]}{(2)} - SS[R \text{ linear}]$$

$$= 1760$$

$$SS[R \text{ quad} \times C] = r\sum_i[\sum_j P_{2j}\overline{y}_{ij.}]^2/\Sigma P_{2j}^2 - SS[R \text{ quad}]$$

$$= \frac{4[(-0.17)^2 + 7.38^2 + (-4.65)^2]}{(6)} - SS[R \text{ quad}]$$

$$= 49$$

Figure 6.4 along with the cell means. The plot illustrates the Rate(linear) by City interaction. The response to rate is linear for each city. The significance of the interaction between city and the linear partition for rate shows up in the plot as a different linear response of zinc to rate for each city.

The linear regression contrasts for rate are simple effects for rate computed for each of the cities. The linear regression lines can be computed for each city from the estimated effects in Table 6.9 or with a standard regression computer program. The regression line can be computed following the procedure in Section 3.3 using the cell means in Table 6.9. The linear orthogonal polynomial coefficient estimate for city A is

$$a_1 = \Sigma P_{1j}\overline{y}_{1j}/\Sigma P_{1j}^2$$

$$= \frac{-1(24.55) + 0(30.45) + 1(36.18)}{[(-1)^2 + 0^2 + 1^2]} = \frac{11.63}{2} = 5.8$$

The mean for city A is $\overline{y}_{1..} = 30.39$. With $\lambda_1 = 1$, $\overline{R} = 1.0$, and $d = 0.5$ (Display 3.5), the transformation to an equation in terms of rate (R) is

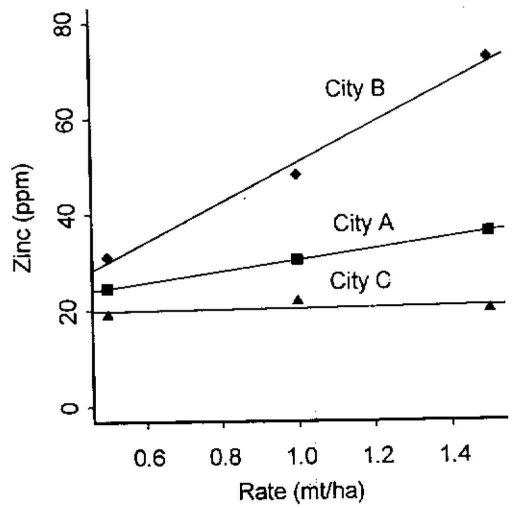

Figure 6.4 Mean zinc content versus rate of sludge application for three cities

$$\widehat{y} = \overline{y}_{1..} + a_1 P_1$$
$$= 30.39 + 5.8\frac{(R-1.0)}{0.5} = 18.79 + 11.6(R)$$

The linear contrast for city C (0.95 in Table 6.9) is considerably smaller than that for the other two cities, 11.63 and 41.42 for cities A and B, respectively. The standard error for the rate linear contrast on the basis of cell means in Table 6.9 for any particular city is

$$s_c = \sqrt{\frac{(MSE)[(-1)^2 + 0^2 + 1^2]}{r}} = \sqrt{\frac{(19.18)[2]}{4}} = 3.1$$

The 95% simultaneous confidence intervals for the three linear contrasts require the Bonferroni $t_{.05,3,27} = 2.55$. The 95% SCI for cities A, B, and C are, respectively, $(3.73, 19.54), (33.52, 49.33)$, and $(-6.96, 8.86)$. The linear responses for cities A and B are significantly positive, with city B having the largest positive linear contrast. The interval for city C includes 0, and we can conclude that zinc will accumulate in barley crops fertilized with increasing amounts of the sludge by-product to the greatest extent from city B and to a lesser extent from that of city A, but that there will be no significant accumulation from that of city C.

Two Quantitative Factors

The response to two quantitative factors can be represented by a polynomial equation with two independent variables. The degree of the polynomial will depend on the number of levels for each of the factors. First-degree equations can represent a factor with two levels, second-degree equations for three levels, and so forth. The geometric representation of a polynomial equation with two independent variables is a response surface in three dimensions. For example, suppose the levels of factor A and factor B are represented by two metrical variables x_1 and x_2 in a quadratic polynomial. The second-degree polynomial equation

$$y = \beta_o + \beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_2 + \beta_4 x_2^2 + \beta_5 x_1 x_2 \qquad \textbf{(6.22)}$$

is an empirical function commonly used for the approximation of a response surface in experimental studies. The quadratic surface can be explored for factor levels that result in the optimum response or different combinations of factor levels with equivalent responses.

The analysis of the factorial experiment with two quantitative factors consists of orthogonal polynomial partitions for the factor main effect and interaction sums of squares. The nature of the polynomial response function can be determined from these partitions. A graph of the responses can be used as an aid to interpret the role that each of the factors plays in the response.

Example 6.4 Water Uptake by Barley Plants

Deposited salts accumulate in soils irrigated for agronomic and horticultural crops. The increased soil salinity eventually suppresses plant development and crop yields.

Research Hypothesis: An investigator hypothesized that exposure of plants to high levels of salts in their media over time eventually inhibits the plant's uptake of water and nutrients from the soil, thus suppressing the growth and development of plants. An experiment was conducted with barley plants to measure the effect of growth medium salinity on water uptake by the plants.

Treatment Design: A factorial arrangement was used with "salinity of media" and "age of plant" in days as the two factors. The plants were grown in nutrient solutions with the salinity level adjusted to three different levels. The salinity levels expressed as units of osmotic pressure were 0, 6, and 12 bars. Plants were harvested at 14, 21, and 28 days.

Experiment Design: Each of the nine treatment combinations of salinity and days was assigned to two replicate containers in a completely randomized design. The containers were placed in the growth chamber in a completely randomized arrangement.

One of the measurements made at harvest was the amount of water uptake by the plants during the experiment. Water uptake is expressed as milliliters of water uptake per 100 grams of plant dry weight. The data are shown in Table 6.10, and the analysis of variance is shown in Table 6.11. Manual calculations for the sums of squares partitions are illustrated in Table 6.12.

Computational Notes: The arrangement in Table 6.12 is convenient for manual computation of the sums of squares partitions from the cell means. Main effect and

Table 6.10 Water uptake (ml/100 g) by barley plants at 14, 21, and 28 days grown in solutions with salinity levels of 0, 6, and 12 bars

Salinity	0 bars			6 bars			12 bars		
Days	14	21	28	14	21	28	14	21	28
	2.2	5.0	13.2	3.7	5.9	9.4	2.8	4.5	7.6
	3.3	5.7	12.4	4.5	7.2	11.0	3.4	5.9	8.3
Means ($\overline{y}_{ij.}$)	2.75	5.35	12.80	4.10	6.55	10.20	3.10	5.20	7.95

Source: Dr. T. C. Tucker, Department of Soil and Water Science, University of Arizona.

Table 6.11 Analysis of variance for water uptake by barley plants

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	17	184.73			
Salinity (S)	2	9.51	4.75	8.52	.008
S linear	1	7.21	7.21	12.92	.006
S quadratic	1	2.30	2.30	4.12	.073
Days (D)	2	151.99	75.99	136.24	.000
D linear	1	147.00	147.00	263.55	.000
D quadratic	1	4.99	4.99	8.94	.015
Salinity × Days (SD)	4	18.21	4.55	8.16	.005
S lin × D lin	1	13.52	13.52	24.24	.001
S lin × D quad	1	2.94	2.94	5.27	.047
S quad × D lin	1	1.21	1.21	2.18	.174
S quad × D quad	1	0.53	0.53	0.96	.353
Error	9	5.02	0.56		

interaction partitions can be computed in the same table. Main effect partitions are normally computed from marginal means. However, cell means are used in Table 6.12 to compute the main effect partitions: thus, repeated values of polynomial contrast coefficients for main effect partitions are necessary for the cell means that contribute to each of their respective marginal means. For example, S_l, the linear contrast for salinity, requires a -1 for each cell at the 0-bar level, a 0 for each cell at the 6-bar level, and a $+1$ for each cell at the 12-bar level.

The coefficients for the interaction partitions are determined as the product of the coefficients for the corresponding components of the interaction. For example, the coefficients for the interaction between salinity(linear) and day(linear) in Table 6.12 are formed as the products of the coefficients for the linear contrast of the main effects for the two factors. Each coefficient for S_lD_l is a product of the corresponding coefficients for S_l and D_l. The computation is exhibited in Display 6.3.

Table 6.12 Calculation of linear and quadratic sums of squares partitions for salinity and day main effect and interaction sums of squares

Bars	*Days*	*Means* ($\overline{y}_{ij.}$)	S_l	S_q	D_l	D_q	S_lD_l	S_lD_q	S_qD_l	S_qD_q
0	14	2.75	−1	1	−1	1	1	−1	−1	1
	21	5.35	−1	1	0	−2	0	2	0	−2
	28	12.80	−1	1	1	1	−1	−1	1	1
6	14	4.10	0	−2	−1	1	0	0	2	−2
	21	6.55	0	−2	0	−2	0	0	0	4
	28	10.20	0	−2	1	1	0	0	−2	−2
12	14	3.10	1	1	−1	1	−1	1	−1	1
	21	5.20	1	1	0	−2	0	−2	0	−2
	28	7.95	1	1	1	1	1	1	1	1
		$\Sigma P_{cij}\overline{y}_{ij.}$	−4.7	−4.6	21.0	6.7	−5.2	−4.2	2.7	3.1
		ΣP_{cij}^2	6.0	18.0	6.0	18.0	4.0	12.0	12.0	36.0
		SS*	7.2	2.3	147.0	5.0	13.5	2.9	1.2	0.5
		Effect†	−0.78	−0.25	3.50	0.37	−1.30	−0.35	0.23	0.09

*$SS = r(\Sigma P_{cij}\overline{y}_{ij.})^2/\Sigma P_{cij}^2$; †$Effect = \Sigma P_{cij}\overline{y}_{ij.}/\Sigma P_{cij}^2$

Display 6.3 Computation of Coefficients for Orthogonal Polynomial Interaction Contrasts

S_l:	−1	−1	−1	0	0	0	1	1	1
D_l:	−1	0	1	−1	0	1	−1	0	1
S_lD_l:	1	0	−1	0	0	0	−1	0	1

Interpretations for the Regression Contrasts

The F_0 statistics in Table 6.11 indicate significant interaction of the salinity(linear) partition with the day(linear) and day(quadratic) partitions. Main effect partitions for salinity(linear), day(linear), and day(quadratic) were also significant. None of the salinity quadratic effects were significant at the .05 level of significance. The response of water uptake to salinity level was the primary focus of the investigation. A profile plot facilitates the interpretation of the results with significant interaction. A plot of the cell means and the linear regression of water uptake on salinity for each day is shown in Figure 6.5.

The linear regression lines for water uptake on salinity computed separately for each day are shown in Figure 6.5 along with the estimated treatment means. The salinity linear contrasts for each day computed from the S_l column in Table 6.12 are shown in Display 6.4 along with their 95% simultaneous confidence intervals using the Bonferroni $t_{.05,3,9} = 2.93$. The 95% SCI indicate salinity had no effect on the water uptake by the plants for the first three weeks up to day 21 since the intervals for 14 and 21 days include 0. However, by the end of the fourth week, day 28,

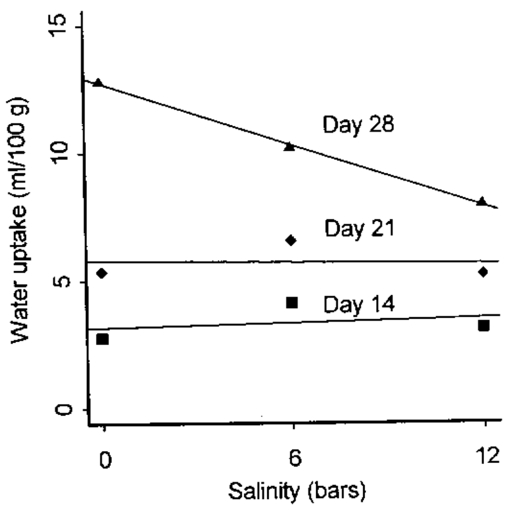

Figure 6.5 Water uptake of barley plants at three salinity levels for 14, 21, and 28 days

Display 6.4 Linear Contrasts for Salinity for Each Day

Day	*Linear Contrast*	*95% SCI* (L, U)
14	$S_l = -1(2.75) + 0(4.10) + 1(3.10) = 0.35$	$(-1.85, 2.55)$
21	$S_l = -1(5.35) + 0(6.55) + 1(5.20) = -0.15$	$(-2.35, 2.05)$
28	$S_l = -1(12.80) + 0(10.20) + 1(7.95) = -4.85$	$(-7.05, -2.65)$
Standard error	$s_c = \sqrt{MSE[(-1)^2 + 0^2 + 1^2]/r} = \sqrt{0.56} = 0.75$	

water uptake by the plants decreased with an increase in the salinity of the medium since the upper limit of the confidence interval for the linear contrast was -2.65.

The differences among the linear responses to salinity resulted in significant interactions between the salinity(linear) partition and the day partitions. The significant interaction between salinity(linear) and day(quadratic) effects indicates that the linear response to salinity changes in a quadratic manner over days.

From day 14 to day 28 the S_l contrast values decrease in a quadratic trend. The decrease in the contrast value from day 14 to day 21 is $(-0.15) - 0.35 = -0.50$, whereas the decrease in the contrast from day 21 to day 28 is $(-4.85) - (-0.15) = -4.70$.

Computing a Response Surface Equation

The polynomial equation relating water uptake to salinity and days will include the terms judged significant in the analysis of variance tests. The significant terms were S_l, D_l, D_q, S_lD_l, and S_lD_q. The equation in terms of S and D will be

$$y = \beta_o + \beta_1 S + \beta_2 D + \beta_3 D^2 + \beta_4 SD + \beta_5 SD^2 \qquad \textbf{(6.23)}$$

The coefficients for Equation (6.23) may be estimated directly from the data with a multiple regression program, or they may be evaluated from the orthogonal polynomial equation as shown in Section 3.3. The polynomial coefficient contrasts in Display 3.5 are used for the transformations. Let P_{cs} and P_{cd} represent the polynomial contrasts for salinity and day, respectively. The orthogonal polynomial equation can be written as

$$y = \mu + \alpha_1 P_{1s} + \gamma_1 P_{1d} + \gamma_2 P_{2d} + (\alpha\gamma)_{11} P_{1s} P_{1d} + (\alpha\gamma)_{12} P_{1s} P_{2d} \qquad \textbf{(6.24)}$$

where α_1 is the linear polynomial coefficient for salinity, γ_1 is the linear coefficient for day, and $(\alpha\gamma)_{12}$ is the interaction coefficient for salinity linear by day quadratic. The estimates of the coefficients in Equation (6.24) are calculated in Table 6.12 as Effect $= \Sigma\, P_{cij}\overline{y}_{ij.} / \Sigma\, P_{cij}^2$. For example, the estimate for $(\alpha\gamma)_{11}$ from the S_lD_l line in Table 6.12 is $-5.2/4 = -1.3$. The term $(\alpha\gamma)_{11} P_{1s} P_{1d}$ in Equation (6.24) with $\lambda_1 = 1$ becomes

$$-1.3\lambda_1 \left(\frac{S-6}{6}\right) \lambda_1 \left(\frac{D-21}{7}\right) = -0.0310(S-6)(D-21)$$

The remaining estimates are computed in the same manner with $\widehat{\mu} = \overline{y}_{...} = 6.44$, $\lambda_1 = 1$, and $\lambda_2 = 3$, and the resulting equation is

$$\widehat{y} = 5.70 - 0.0133(-6) + 0.50(D-21) + 0.0227(D-21)^2$$
$$-0.0310(S-6)(D-21) - 0.0036(S-6)(D-21)^2$$

These equations can be used to explore response surfaces for maxima or minima, or they may be used to determine values of the factors that result in equivalent responses. Specialized tools for these methods are discussed later in Chapter 13, Response Surface Designs.

6.6 Three Treatment Factors

The inclusion of additional factors in the treatment design increases the complexity of interaction patterns among the treatment factors. The number of treatment combinations increases rapidly as factors are added to the design. The three-factor design with a levels of A, b levels of B, and c levels of C has abc treatment combinations. A fourth factor, D, with d levels further increases the number of treatments by a multiple of d.

The two-factor design enables the investigation of the *first-order,* or two-factor, interaction AB. The two additional first-order interactions, AC and BC, in the three-factor design broaden the inferences from the study. In addition, there is a *second-order,* or three-factor, interaction, ABC, to consider. *Third-order,* or four-factor, interactions such as $ABCD$ enter the increasingly complex interaction inference structure as factors are added to the design.

Example 6.5 Shrimp Culture in Aquaria

The California brown shrimp spawn at sea and the hatched eggs undergo larval transformations while being transported toward the shore. By the time they transform to post-larval stage they enter estuaries, where they grow rapidly into subadults and migrate back offshore as they approach sexual maturity.

The shrimp encounter wide temperature and salinity variation in their life cycle as a result of their migrations during the cycle. Thus, a knowledge of how temperature and salinity affect their growth and survival is of great importance to understanding their life history and ecology.

There was at the time of this experiment great interest in commercial culture of the shrimp. From the standpoint of mariculture another important factor was stocking density in the culture tanks that affects intraspecific competition.

Research Objective: The investigators wanted to know how water temperature, water salinity, and density of shrimp populations influenced the growth rate of shrimp raised in aquaria and whether the factors acted independently on the shrimp populations.

Treatment Design: A factorial arrangement was used with three factors: "temperature" (25°C, 35°C); "salinity" of the water (10%, 25%, 40%); and "density" of shrimp in the aquarium (80 shrimp/40 liters, 160 shrimp/40 liters). The levels chosen were those considered most likely to exhibit an effect if the factor was influential on shrimp growth.

Experiment Design: The experiment design consisted of three replicate aquaria for each of the 12 treatment combinations of the $2 \times 2 \times 3$ factorial. Each of the 12 treatment combinations was randomly assigned to three aquaria for a completely randomized design. The 36 aquaria were stocked with post-larval shrimp at the beginning of the test. The weight gain of the shrimp in four weeks for each of the 36 aquaria is shown in Table 6.13 on a per-shrimp basis.

Table 6.13 Four-week weight gain of shrimp cultured in aquaria at different levels of temperature (T), density of shrimp populations (D), and water salinity (S)

T	D	S	*Weight Gain (mg)*
25°C	80	10%	86, 52, 73
		25%	544, 371, 482
		40%	390, 290, 397
	160	10%	53, 73, 86
		25%	393, 398, 208
		40%	249, 265, 243
35°C	80	10%	439, 436, 349
		25%	249, 245, 330
		40%	247, 277, 205
	160	10%	324, 305, 364
		25%	352, 267, 316
		40%	188, 223, 281

Source: Dr. J. Hendrickson and K. Dorsey, Department of Ecology and Evolutionary Biology, University of Arizona.

The Statistical Model for Three Factors

The cell means model for a three-factor experiment with r replications of each of the abc treatment combinations in a completely randomized design is

$$y_{ijkl} = \mu_{ijk} + e_{ijkl} \tag{6.25}$$

$$i = 1, 2, \ldots, a \quad j = 1, 2, \ldots, b \quad k = 1, 2, \ldots, c \quad l = 1, 2, \ldots, r$$

The cell mean μ_{ijk} expressed as a function of the factorial main effects and interactions is

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} \tag{6.26}$$

where $\mu = \overline{\mu}_{...}$ is the general mean and α_i, β_j, and γ_k are the main effects of A, B, and C. The respective two-factor interaction effects are $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ik}$, and $(\beta\gamma)_{jk}$; and the three-factor interaction effect is $(\alpha\beta\gamma)_{ijk}$. The definitions of main effects and two-factor interactions follow from the derivations given in Equations (6.6) to (6.8) for the two-factor experiment. The main effects are

$$\alpha_i = (\overline{\mu}_{i..} - \overline{\mu}_{...}),\ \beta_j = (\overline{\mu}_{.j.} - \overline{\mu}_{...}),\ \gamma_k = (\overline{\mu}_{..k} - \overline{\mu}_{...})$$

and a typical two-factor interaction is

$$\begin{aligned}(\beta\gamma)_{jk} &= (\overline{\mu}_{.jk} - \overline{\mu}_{...}) - \beta_j - \gamma_k \\ &= \overline{\mu}_{.jk} - \overline{\mu}_{.j.} - \overline{\mu}_{..k} + \overline{\mu}_{...}\end{aligned} \tag{6.27}$$

The three-factor interaction occurs when the main effects and two-factor interactions do not satisfactorily explain the variation in the cell mean deviations $(\mu_{ijk} - \overline{\mu}_{...})$. The three-factor interaction is the difference between the cell mean deviation and the sum of the main effects and two-factor interaction effects:

$$\begin{aligned}(\alpha\beta\gamma)_{ijk} &= (\mu_{ijk} - \overline{\mu}_{...}) - [\alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk}] \\ &= \mu_{ijk} - \overline{\mu}_{ij.} - \overline{\mu}_{i.k} - \overline{\mu}_{.jk} + \overline{\mu}_{i..} + \overline{\mu}_{.j.} + \overline{\mu}_{..k} - \overline{\mu}_{...}\end{aligned} \tag{6.28}$$

The Analysis for Three Factors

The sum of squares for treatments is partitioned into main effect and interaction sums of squares as

$$\begin{aligned} SS\text{ Treatment} &= SS\boldsymbol{A} + SS\boldsymbol{B} + SS\boldsymbol{C} + SS(\boldsymbol{AB}) \\ &\quad + SS(\boldsymbol{AC}) + SS(\boldsymbol{BC}) + SS(\boldsymbol{ABC}) \end{aligned} \tag{6.29}$$

Keep in mind that the degrees of freedom for main effect sums of squares are $(a-1)$, $(b-1)$, and $(c-1)$, respectively, for factors A, B, and C. The degrees of freedom for two-factor interactions are the product of the main effect degrees of freedom for the included factors. Likewise, the degrees of freedom for a three-factor or higher interaction are the product of the main effect degrees of freedom for the included factors, so that $SS(\boldsymbol{ABC})$ has $(a-1)(b-1)(c-1)$ degrees of freedom.

The sums of squares partitions and analysis of variance table for the three-factor shrimp growth experiment are shown in Table 6.14. The Mean Square for Error in Table 6.14 is the denominator of the F_0 statistic to test the null hypothesis for any set of factorial effects with the fixed effects model. The F_0 statistics in Table 6.14 lead to the rejection of the null hypothesis for the TS two-factor interaction, the *TDS* three-factor interaction, and all main effects. The cell and marginal means for all factors are shown in Table 6.15.

Table 6.14 Analysis of variance for weight gain of shrimp cultured in aquaria

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	35	537,327.01			
Temp (*T*)	1	15,376.00	15,376.00	5.30	.030
Salinity (*S*)	2	96,762.50	48,381.25	16.66	.000
Density (*D*)	1	21,218.78	21,218.78	7.31	.012
TS	2	300,855.17	150,427.58	51.80	.000
TD	1	8,711.11	8,711.11	3.00	.096
SD	2	674.39	337.19	0.12	.891
TDS	2	24,038.39	12,019.19	4.14	.029
Error	24	69,690.67	2,903.78		

Table 6.15 Cell and marginal means of four-week weight gain of shrimp cultured in aquaria at different levels of temperature (T), density of shrimp populations (D), and water salinity (S)

Cell Means ($\overline{y}_{ijk.}$)

	Density 80		*Density* 160		
	Temperature		*Temperature*		
Salinity	25°	35°	25°	35°	$\overline{y}_{..k.}$
10%	70	408	71	331	220
25%	466	275	333	312	346
40%	359	243	252	231	271
T × D Means ($\overline{y}_{ij..}$)	298	309	219	291	

T × S Means ($\overline{y}_{i.k.}$)

T	*S* 10%	25%	40%	$\overline{y}_{i...}$
25°	71	399	306	259
35°	370	293	237	300

D × S Means ($\overline{y}_{.jk.}$)

D	*S* 10%	25%	40%	$\overline{y}_{.j..}$
80	239	370	301	303
160	201	322	242	255

Interpretations must be conditioned on some measure of statistical significance in conjunction with the biological significance of the responses. Standard errors of cell and marginal means are required for any subsequent statistical tests of specific comparisons. The standard error for any mean is $s_{\overline{y}} = \sqrt{MSE/n}$, where n is the number of observations in the mean. The standard error of the difference between any pair of means is $s(\overline{y}_i - \overline{y}_j) = \sqrt{2MSE/n}$. The estimated standard errors for the shrimp culture experiment are shown in Table 6.16.

Some Preliminary Interpretations About the Factor Effects

The significance of the three-factor interaction indicates that temperature, salinity, and density are interrelated in their effect on the shrimp growth. The significant three-factor interaction implies that the interaction between two factors is not constant over levels of the third factor. Consider the interaction between density and salinity separately at temperatures of 25°C and 35°C, as shown in the graphs of cell means in Figure 6.6.

A comparison of the simple effects of salinity at each level of density and temperature can be used to interpret the results. The simple effects of salinity are best estimated as orthogonal polynomial linear and quadratic contrasts at each combination of density and temperature. Sums of squares partitions computed for the three factor interaction *TDS* are $SS(T \times D \times S\ linear) = 11{,}051$ and $SS(T \times D \times S\ quadratic) = 12{,}987$ with P-values .063 and .045, respectively, indicating the salinity quadratic coefficient is dependent on the levels of temperature and density.

Table 6.16 Standard errors for cell and marginal means in a three-factor treatment design

Temperature: $a = 2$ *levels*; *Density*: $b = 2$ *levels*; *Salinity*: $c = 3$ *levels*

Main Factor Means		
Temperature	*Salinity*	*Density*
$\sqrt{\frac{MSE}{rbc}} = \sqrt{\frac{2903.78}{18}} = 12.7$	$\sqrt{\frac{MSE}{rab}} = \sqrt{\frac{2903.78}{12}} = 15.6$	$\sqrt{\frac{MSE}{rac}} = \sqrt{\frac{2903.78}{18}} = 12.7$

Two Factor Marginal Means	
Density by Temperature	*Density by Salinity*
$\sqrt{\frac{MSE}{rc}} = \sqrt{\frac{2903.78}{9}} = 18.0$	$\sqrt{\frac{MSE}{ra}} = \sqrt{\frac{2903.78}{6}} = 22.0$
Salinity by Temperature	*Cell Means*
$\sqrt{\frac{MSE}{rb}} = \sqrt{\frac{2903.78}{6}} = 22.0$	$\sqrt{\frac{MSE}{r}} = \sqrt{\frac{2903.78}{3}} = 31.1$

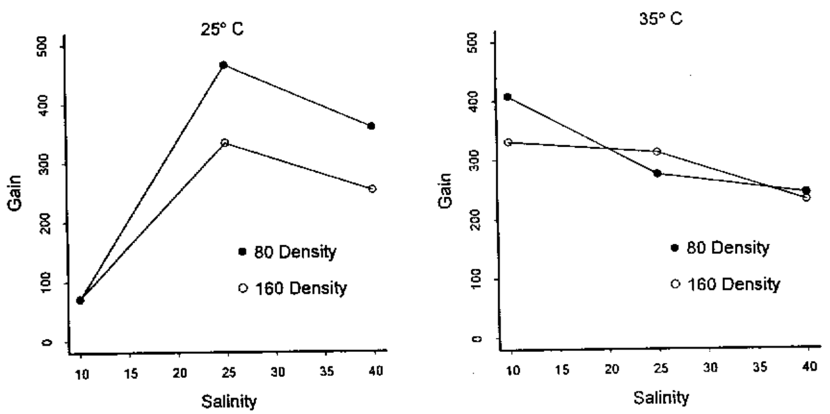

Figure 6.6 Weight gain of shrimp cultured in aquaria in a $2 \times 2 \times 3$ factorial arrangement of temperature, density, and salinity

The salinity quadratic coefficients were computed as orthogonal polynomial contrasts for the four combinations of temperature and density from the cell means in Table 6.15 following a template similar to that provided in Table 6.12. For example, the quadratic coefficient for salinity at 25°C and a density of 80 is

$$\frac{[70 - 2(466) + 359]}{[1^2 + (-2^2) + 1^2]} = -\frac{503}{6} = -83.8$$

with a standard error $\sqrt{2903.78/6(3)} = 12.7$. The 95% SCI estimates were computed for the four coefficients based on the Bonferroni $t_{.05,4,24} = 2.70$.

The 95% SCI estimates of the quadratic coefficients for salinity at 25°C are $(-118.1, -49.5)$ for a density of 80 and $(-91, -22.9)$ for a density of 160, while the estimates at 35°C are $(-17.5, 51.1)$ for a density of 80 and $(-44.6, 24.0)$ for a density of 160. Clearly the quadrature at 25°C is significant since the 95% SCI do not include 0 and not significant at 35°C since those intervals include 0.

6.7 Estimation of Error Variance with One Replication

Situations arise in research studies wherein only one observation is available in each cell of a factorial arrangement. The experimental error variance cannot be estimated with only one replication of the treatment combinations. The sums of squares partitions for factor main effects and interaction are equal to the total sum of squares for the observations.

Additivity describes the case when there is no interaction between factors. Under additivity of factors the mean square partition for interaction can be used as an estimate of experimental error. The additivity of main effects or absence of interaction is not guaranteed, and some means of evaluating the presence of interaction is required.

Error Variance Estimates with Two Quantitative Factors

The additivity of quantitative factors can be investigated with the interaction components for linear and possibly quadratic regression partitions (Section 6.5). For example, the sum of squares for linear $\times$ linear interaction can be partitioned out of the interaction sum of squares with the assumption that the remaining sum of squares for deviations from linear $\times$ linear interaction is experimental error. These sums of squares for deviations from the linear $\times$ linear interaction would include all higher orders of polynomial interaction such as linear $\times$ quadratic, and so forth. The mean square for deviations from the linear $\times$ linear interaction can be used as the mean square for error. The number of 1 degree of freedom interaction terms that are partitioned from the interaction is a matter of judgment based on the number of degrees of freedom available for a reasonably powerful test for interaction and main effects.

Error Variance Estimates with a Qualitative and a Quantitative Factor

The same approach can be used if one of the factors is qualitative and the other is quantitative. In this case, the sum of squares for the interaction between the qualitative factor and the linear effect of the quantitative factor can be partitioned out of the interaction sum of squares (Section 6.5). The remaining deviations' sum of squares can be used to estimate experimental error.

Error Variance Estimates with Two Qualitative Factors

If both factors are qualitative the problem is somewhat more difficult. Tukey (1949b) gave a method for isolating a 1 degree of freedom sum of squares to test for nonadditivity in a two-way classification with one observation per cell. The term for nonadditivity in the linear model is a simple product of the main effects, $\lambda\alpha_i\beta_j$, where the parameter λ represents the added parameter for nonadditivity. The product of main effects is a multiplicative form of interaction, and if there is nonadditivity from this specific type of interaction between the main effects, α_i and β_j, then $\lambda \neq 0$. Under this model the cell means are a sum of the general mean, the factor effects, and the product term, or

$$\mu_{ij} = \overline{\mu}_{..} + (\overline{\mu}_{i.} - \overline{\mu}_{..}) + (\overline{\mu}_{.j} - \overline{\mu}_{..}) + \lambda(\overline{\mu}_{i.} - \overline{\mu}_{..})(\overline{\mu}_{.j} - \overline{\mu}_{..})$$

The sum of squares for nonadditivity requires a computation involving the deviations of the A and B means from the grand mean, $(\overline{y}_{i.} - \overline{y}_{..})$ and $(\overline{y}_{.j} - \overline{y}_{..})$, respectively. The technique is illustrated with Example 6.6.

Example 6.6 Hearing Levels in Adult Males

The data in Table 6.17 are the percentage of men aged 55 to 64 with hearing levels 16 decibels above the audio metric zero. The row categories were sound levels in cycles per second (hertz), and the column categories were seven occupational categories.

The required computations include

$$P_j = \sum_{i=1}^{a} (\overline{y}_{i.} - \overline{y}_{..})y_{ij} \tag{6.30}$$

$$P = \sum_{j=1}^{b} P_j(\overline{y}_{.j} - \overline{y}_{..}) = 17{,}416 \tag{6.31}$$

and

$$\sum_{i=1}^{a} (\overline{y}_{i.} - \overline{y}_{..})^2 = 6{,}944, \qquad \sum_{j=1}^{b} (\overline{y}_{.j} - \overline{y}_{..})^2 = 162 \tag{6.32}$$

The 1 degree of freedom sum of squares for nonadditivity is

Table 6.17 Percentage of men with hearing levels 16 decibels above the audio metric zero classified in a 7 × 7 factorial arrangement with one observation per cell

	B								
A	1	2	3	4	5	6	7	$\overline{y}_{i.}$	$(\overline{y}_{i.} - \overline{y}_{..})$
1	2.1	6.8	8.4	1.4	14.6	7.9	4.8	6.6	− 31.6
2	1.7	8.1	8.4	1.4	12.0	3.7	4.5	5.7	− 32.5
3	14.4	14.8	27.0	30.9	36.5	36.4	31.4	27.3	− 10.9
4	57.4	62.4	37.4	63.3	65.5	65.6	59.8	58.8	20.6
5	66.2	81.7	53.3	80.7	79.7	80.8	82.4	75.0	36.8
6	75.2	94.0	74.3	87.9	93.3	87.8	80.5	84.7	46.5
7	4.1	10.2	10.7	5.5	18.1	11.4	6.1	9.4	− 28.8
$\overline{y}_{.j}$	31.6	39.7	31.4	38.7	45.7	41.9	38.5		$\overline{y}_{..} = 38.2$
$(\overline{y}_{.j} - \overline{y}_{..})$	− 6.6	1.5	− 6.8	0.5	7.5	3.7	0.3		
P_j	6719	7730	5046	7776	6850	7313	7192		

Source: C. Daniel (1978), Patterns in residuals in the two-way layout. *Technometrics* 20, 385–395. Data originally published in J. Roberts and J. Cohrssen (1968), *Hearing Levels of Adults*, Table 4, p. 36. U.S. National Center for Health Statistics Publications, Series 11, No. 31. Rockland, Md.

$$S(\text{Nonadditivity}) = \frac{P^2}{\sum_{i=1}^{a} (\overline{y}_{i.} - \overline{y}_{..})^2 \sum_{j=1}^{b} (\overline{y}_{.j} - \overline{y}_{..})^2} = \frac{17{,}416^2}{(6{,}944)(162)} = 269.6 \tag{6.33}$$

The analysis of variance for the data is shown in Table 6.18 with the sum of squares for error partitioned into a 1 degree of freedom sum of squares for nonadditivity and a residual sum of squares.

Table 6.18 One degree of freedom partition for nonadditivity in the analysis of variance for a 7 × 7 factorial with one observation per cell

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*
Rows	6	48,589.1	8,098.2
Columns	6	1,141.5	190.2
Error	36	1,444.7	40.1
Nonadditivity	1	269.6	269.6
Residual	35	1,175.1	33.6

The null hypothesis of no nonadditivity is tested with the statistic $F_0 = MS(\text{Nonadditivity})/MS(\text{Residual}) = 269.6/33.6 = 8.02$. The null hypothesis is rejected with a critical region $F_0 > F_{.05,1,35} = 4.12$.

Several methods have been developed to ascertain the source of nonadditivity in a two-way table. Daniel (1978) used a method based on the residuals in each of the cells, $y_{ij} - \overline{y}_{i.} - \overline{y}_{.j} + \overline{y}_{..}$. Technical discussions and examples of several other methods and models for nonadditivity can be found in Johnson and Graybill (1972), Bradu and Gabriel (1978), and Mandel (1971).

6.8 How Many Replications to Test Factor Effects?

Procedures were given in Section 2.14 to estimate replication numbers based on the test for differences among treatment means with the F_0 statistic. The values of Φ (Equation (2.25)) can be applied directly to a test for differences among cell means in the factorial arrangement with the null hypothesis H_0: $\mu_{11} = \mu_{12} = \cdots = \mu_{ab}$. In this case, the factorial structure is ignored and the cell means model $y_{ijk} = \mu_{ij} + e_{ijk}$ expressed in the effects model form is $y_{ijk} = \mu + \tau_{ij} + e_{ijk}$, where τ_{ij} is the effect of the ijth treatment combination in the factorial arrangement. Then

$$\Phi^2 = \frac{r\sum_{i=1}^{a}\sum_{j=1}^{b}\tau_{ij}^2}{ab\sigma^2} \tag{6.34}$$

is used to estimate replication numbers from the charts based on the values of τ_{ij} required to be significant.

If replication numbers based on the factorial effects are required, the noncentrality parameters are

$$\lambda_a = br\sum_{i=1}^{a}\frac{\alpha_i^2}{\sigma^2}, \quad \lambda_b = ar\sum_{j=1}^{b}\frac{\beta_j^2}{\sigma^2}, \quad \text{and} \quad \lambda_{ab} = r\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{(\alpha\beta)_{ij}^2}{\sigma^2} \tag{6.35}$$

respectively, for A and B main effects and AB interaction. Then Φ is determined as $\Phi = \sqrt{\lambda/(\nu_1 + 1)}$, where ν_1 are the numerator degrees of freedom for the F_0 statistic.

6.9 Unequal Replication of Treatments

Missing data in research studies is inevitable. The design is no longer balanced with a complete data set, and standard computing formulae no longer apply. Before the advent of modern computing, a complete data set was most advantageous because relatively simple formulae could be used for manual computations. Much effort was put into developing methods for the analysis of variance sum of squares partitions when there were unequal numbers of observations among the cells of the

factorial arrangement. General statistical routines programmed to accommodate known statistical theory have removed the computational burdens associated with the analysis of incomplete data sets.

Orthogonality Is Lost with Missing Observations

The sum of squares for one factorial effect will convey some information about other factorial effects when there are unequal numbers of observations for the treatment combinations and the sum of squares partitions are computed in the usual manner. This non-orthogonal relationship in the sum of squares partitions for the analysis of variance requires us to consider carefully the estimates we use for parameters in the model and the statistics we use to test the critical hypotheses in an analysis of the study.

Orthogonal contrasts were introduced in Chapter 3 as contrasts that do not convey information about one another. Orthogonality carries the same meaning with observations in a factorial treatment design. When there are equal numbers of observations on each treatment combination, the sums of squares in the analysis of variance constitute an orthogonal partition of the treatment sum of squares. In Section 6.4 the additive partition of SS Treatment for a balanced two-factor experiment was

$$SS\ \text{Treatment} = SSA + SSB + SS(AB)$$

and the sum of squares for one factorial effect did not convey any information about other factorial effects.

The example data shown in Display 6.5 illustrate the complications introduced by unequal treatment replication in a factorial treatment arrangement. The data in the cells represent the average speed in excess of the posted speed limit traveled by automobiles involved in 20 fatal accidents, 10 occurring in rainy weather and 10 occurring in clear weather. The factors for the study are W (Weather) and R (Type of Roadway). Notice that 8 of the 10 accidents in rainy weather were on interstate highways, whereas 8 of the 10 accidents in clear weather were on two-lane highways.

Observation of the marginal means for weather indicates the average speed in excess of the speed limit for rainy weather was slightly higher than that for clear weather, $\overline{y}_{1..} = 13$ versus $\overline{y}_{2..} = 12$. However, the observed cell means indicate an entirely different result. The average speeds in fatal accidents in clear weather were greater by 5 miles per hour than in rainy weather for both interstate $(20 - 15)$ and two-lane $(10 - 5)$ highways.

The unequal treatment replications lead to contradictory results from cell means and marginal means. An excess of accidents occurred on the interstates in rainy weather and an excess of accidents occurred on the two-lane highways in clear weather. Thus, the marginal mean for clear weather is biased downward by the excess of accidents on the two-lane highways with overall slower speeds, and

Display 6.5 Unequal Treatment Replication in a 2 × 2 Factorial for Speeds in Excess of Posted Speed Limit

	Cell Means			
	Interstate	*Two-Lane*	*Sums*	*Means*
Rainy	15	5	130	13
	$r_{11} = 8$	$r_{12} = 2$	$r_{1.} = 10$	
Clear	20	10	120	12
	$r_{21} = 2$	$r_{22} = 8$	$r_{2.} = 10$	
Sums	160	90	250	
	$r_{.1} = 10$	$r_{.2} = 10$	$r_{..} = 20$	$\bar{y}_{..} = 12.5$
Means	16	9		

the marginal mean for rainy weather is biased upward by the excess of accidents on the interstates with overall faster speeds.

The sum of squares for treatments with unequal replication numbers is computed correctly as

$$\text{SS Treatment} = 8(15 - 12.5)^2 + 2(5 - 12.5)^2 + 2(20 - 12.5)^2 + 8(10 - 12.5)^3$$

$$= 325$$

The sums of squares for main effects and interaction computed *incorrectly* with methods outlined in Section 6.4 are

$$SSW = 10(13 - 12.5)^2 + 10(12 - 12.5)^2 = 5$$

$$SSR = 10(16 - 12.5)^2 + 10(9 - 12.5)^2 = 245$$

and

$$SS(WR) = SS \text{ Treatment} - SSW - SSR = 325 - 5 - 245 = 75$$

Inspection of the cell means and the sum of squares for interaction indicate another contradiction in the usual analysis methods. The calculated sum of squares, $SS(WR) = 75$, indicates some interaction is present in the study. However, inspection of the cell means gives no indication of interaction. The observed simple effect of weather is equal to 5 for both highway types. The sum of squares for interaction would be 0 if the sum of squares partitions were computed correctly.

The general principles for a correct analysis of factorial treatment designs with unequal treatment replication are illustrated with the two-factor experiment on durability of asphaltic concrete.

Example 6.7 Asphaltic Concrete Durability Revisited Again

The experiment on tensile strength of asphaltic concrete specimens in Example 6.2 is used to illustrate the analysis of a factorial treatment design with unequal replication of the treatments. For illustration, suppose specimens were constructed with basalt or silicious aggregate types for the three kneading compaction methods. Suppose some of the specimens were damaged prior to testing, resulting in an unequal number of specimens being available among the treatments for tensile strength tests. The data with unequal replication numbers are shown in Table 6.19.

Table 6.19 Tensile strength (psi) of asphaltic concrete specimens for two aggregate types with each of three kneading compaction methods

	Compaction Method			
	Kneading			
Aggregate Type	*Regular*	*Low*	*Very Low*	*Aggregate Means* ($\bar{y}_{i..}$)
Basalt	106	93	56	
	108	101		
		98		
Means ($\bar{y}_{1j.}$)	107.0	97.3	56	93.7
Silicious	107	63	40	
	110	60	41	
	116		44	
Means ($\bar{y}_{2j.}$)	111.0	61.5	41.7	72.6
Compaction means ($\bar{y}_{.j.}$)	109.4	83.0	45.3	

Establish Estimators with the Cell Means Model

The cell means model can be used to establish the appropriate estimators for population parameters and hypotheses that we are required to test. The cell means model is

$$y_{ijk} = \mu_{ij} + e_{ijk} \qquad (6.36)$$

$$i = 1, 2, \ldots, a \quad j = 1, 2, \ldots, b \quad k = 1, 2, \ldots, r_{ij}$$

where μ_{ij} is the cell mean for the ith level of factor A and the jth level of factor B, e_{ijk} is the normally distributed random independent experimental error with mean 0 and variance σ^2, and r_{ij} is the number of replicate observations in cell (ij). We will assume there is at least one observation in each cell of the factorial arrangement, so that $r_{ij} > 0$ for all i and j.

Least Squares Estimators for Cell and Marginal Means

The cell means can be estimated by the least squares method following the procedures outlined in Chapter 2 for the cell means model. The estimators of the cell means are the observed cell means

$$\widehat{\mu}_{ij} = \frac{1}{r_{ij}} \sum_{k=1}^{r_{ij}} y_{ijk} = \overline{y}_{ij.} \tag{6.37}$$

and the estimator for experimental error variance is

$$\widehat{\sigma}^2 = s^2 = \frac{1}{N - ab} \Sigma (y_{ijk} - \overline{y}_{ij.})^2 \tag{6.38}$$

where $N = \Sigma r_{ij.}$.

The estimates of the cell means for the tensile strength of the asphalt concrete specimens are shown in Table 6.19, and the estimate of experimental error is

$$s^2 = \frac{1}{14-6}[(106 - 107.0)^2 + \cdots + (44 - 41.7)^2] = \frac{89.83}{8} = 11.23$$

The unbiased least squares estimators of the marginal means are

$$\widehat{\overline{\mu}}_{i.} = \frac{1}{b}\sum_{j=1}^{b} \widehat{\mu}_{ij} \quad \text{and} \quad \widehat{\overline{\mu}}_{.j} = \frac{1}{a}\sum_{i=1}^{a} \widehat{\mu}_{ij} \tag{6.39}$$

with standard error estimators

$$s_{\widehat{\overline{\mu}}_{i.}} = \sqrt{\frac{s^2}{b^2}\sum_{j=1}^{b}\frac{1}{r_{ij}}} \quad \text{and} \quad s_{\widehat{\overline{\mu}}_{.j}} = \sqrt{\frac{s^2}{a^2}\sum_{i=1}^{a}\frac{1}{r_{ij}}} \tag{6.40}$$

The least squares estimates of the marginal means for the asphalt concrete specimens and their standard error estimates are shown in Table 6.20.

For example, the least squares estimate of the marginal mean for the basalt aggregate type is

$$\widehat{\overline{\mu}}_{1.} = \frac{1}{3}(107.0 + 97.3 + 56.0) = 86.8$$

with standard error estimate

$$s_{\widehat{\overline{\mu}}_{1.}} = \sqrt{\frac{11.23}{3^2}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{1}\right)} = 1.51$$

The observed marginal means, $\overline{y}_{i..}$ and $\overline{y}_{.j.}$, shown in Table 6.19 do not have the same value as the least squares estimates of the marginal means in Table 6.20. The observed marginal means estimate weighted functions of the population means where the weights are proportional to the number of replications in the cells. The expected values of the observed marginal means are

$$E(\overline{y}_{i..}) = \frac{1}{r_{i.}}\sum_{j=1}^{b} r_{ij}\mu_{ij}$$

and

$$E(\overline{y}_{.j.}) = \frac{1}{r_{.j}}\sum_{i=1}^{a} r_{ij}\mu_{ij} \tag{6.41}$$

If the number of observations in the treatment cells of the study is proportional to the frequency with which those treatment combinations occur in the population, then the observed marginal means provide the appropriate estimators for the marginal means in the population. The proportional relationship of observation numbers to population frequencies is common in sample surveys. However, the proportional relationship would not be expected to hold for a designed experiment or comparative observational study, and the least squares estimates in Table 6.20 should be used.

Table 6.20 Least squares estimates of the marginal means for tensile strength of asphalt concrete specimens and their standard errors

	Mean	Standard Error
Aggregate	$\widehat{\overline{\mu}}_{i.}$	$s_{\widehat{\overline{\mu}}_{i.}}$
Basalt	86.8	1.51
Silicious	71.4	1.21
Compaction	$\widehat{\overline{\mu}}_{.j}$	$s_{\widehat{\overline{\mu}}_{.j}}$
Regular	109.0	1.53
Low	79.4	1.53
Very low	48.8	1.93

Hypotheses Unchanged by Unequal Treatment Replication

The hypotheses of interest in the factorial treatment design with unequal replication numbers are unchanged from those of interest with equal replication numbers. The initial research question in the factorial treatment design considers the existence of interaction between factors A and B. The interaction effect measures the differences between the simple effects of A at different levels of B. The difference between levels i and k of A at levels j and m of B is the general form of interaction:

$$(\mu_{ij} - \mu_{kj}) - (\mu_{im} - \mu_{km}) = \mu_{ij} - \mu_{kj} - \mu_{im} + \mu_{km}$$

The null hypothesis of no interaction can be expressed in terms of the cell means as

$$H_0\colon\ \mu_{ij} - \mu_{kj} - \mu_{im} + \mu_{km} = 0 \quad \text{for all } i, j, k, \text{ and } m \tag{6.42}$$

In the absence of interaction, the effect of the individual factors on the response variable can be explored separately with tests of hypotheses about the marginal means. The null hypothesis of interest for factor A is the equality of the marginal means, or

$$H_0\colon\ \overline{\mu}_{1.} = \overline{\mu}_{2.} = \cdots = \overline{\mu}_{a.} \tag{6.43}$$

and that for factor B is

$$H_0\colon\ \overline{\mu}_{.1} = \overline{\mu}_{.2} = \cdots = \overline{\mu}_{.b} \tag{6.44}$$

Weighted Squares of Means for Tests of Hypotheses

Among the many methods put forth for analyzing factorial experiments with unequal replication only the method of *weighted squares of means* proposed by Yates (1934) provides the sum of squares partitions to test all three hypotheses in Equations (6.42) through (6.44). A description of other methods and the hypotheses that can be tested with them can be found in Speed, Hocking, and Hackney (1978). The tests for equality of marginal means are of interest only in the absence of interaction.

Computing Sum of Squares for Interaction from Full and Reduced Models

The sum of squares partition for interaction is determined from the principle of full and reduced models introduced in Chapter 2. The full model expressed in terms of the factorial effects is

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \tag{6.45}$$

The solutions obtained from the least squares normal equations are used to compute the sum of squares of experimental error for the full model as

$$SSE_f = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r_{ij}} [y_{ijk} - \widehat{\mu} - \widehat{\alpha}_i - \widehat{\beta}_j - (\widehat{\alpha\beta}_{ij})]^2 \tag{6.46}$$

Under the null hypothesis of no interaction the reduced model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk} \tag{6.47}$$

The solutions obtained from the least squares normal equations are used to compute the sum of squares of experimental error for the reduced model as

$$SSE_r = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r_{ij}} [y_{ijk} - \widehat{\mu} - \widehat{\alpha}_i - \widehat{\beta}_j]^2 \tag{6.48}$$

The sum of squares for interaction is computed as

$$SS(\boldsymbol{AB}) = SSE_r - SSE_f \tag{6.49}$$

The mean square for experimental error is $MSE = (SSE_f)/(N - ab)$, and the mean square for interaction is $MS(\boldsymbol{AB}) = SS(\boldsymbol{AB})/(a-1)(b-1)$. The usual F_0 statistic, $F_0 = MS(\boldsymbol{AB})/MSE$, tests the null hypothesis of no interaction in Equation (6.42). The calculations are illustrated in Appendix 6A.

Weighted Squares of Means to Test Equality of Main Effects in the Absence of Interaction

Tests of hypotheses can be conducted for the equality of marginal means of the factors if the test for interaction is not significant and it can safely be assumed there is no interaction. The correct sum of squares partitions for the weighted squares of means method to test the null hypotheses in Equations (6.43) and (6.44) are shown in Display 6.6. The analysis is based on the sums of squares of the cell means designated as the observations $x_{ij} = \overline{y}_{ij.}$.

Display 6.6 Weighted Squares of Means Sum of Squares Partitions

Factor A	Factor B
$SS\boldsymbol{A}_w = \sum_i^a w_i(\overline{x}_{i.} - \overline{x}_{[1]})^2$	$SS\boldsymbol{B}_w = \sum_j^b v_j(\overline{x}_{.j} - \overline{x}_{[2]})^2$
$w_i = \left[\frac{1}{b^2}\sum_{j=1}^{b}\frac{1}{r_{ij}}\right]^{-1}$	$v_j = \left[\frac{1}{a^2}\sum_{i=1}^{a}\frac{1}{r_{ij}}\right]^{-1}$
$\overline{x}_{[1]} = \sum_i^a w_i\overline{x}_{i.} \big/ \sum_i^a w_i$	$\overline{x}_{[2]} = \sum_j^b v_j\overline{x}_{.j} \big/ \sum_j^b v_j$
$H_0\colon\ \overline{\mu}_{1.} = \overline{\mu}_{2.} = \cdots = \overline{\mu}_{a.}$	$H_0\colon\ \overline{\mu}_{.1} = \overline{\mu}_{.2} = \cdots = \overline{\mu}_{.b}$

The sum of squares partitions required by the weighted squares of means can be computed by many statistical programs. However, the programs may have several options for the type of sum of squares partitions that are computed for the analysis. It is important that the correct options be used for the programs so that the correct sum of squares is computed by the program.[1]

[1] Programs used for analysis of variance will provide the correct sum of squares for the weighted squares of means. Instructions on the use of most programs will indicate if and how different types of sum of squares partitions can be obtained. The correct sum of squares options for several well known programs are

Program	*Sum of Squares*
SAS GLM	Type III
SPSS MANOVA	UNIQUE
MINITAB GLM	Adjusted
BMDP 2V	Default
Splus	summary.aov(...,ssType=3)

The analysis of variance for the asphaltic concrete specimens tensile strengths in Example 6.7 is shown in Table 6.21.

Table 6.21 Analysis of variance for tensile strength of asphaltic concrete specimens with unequal replication of treatments with Yates' weighted squares of means

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	13	10,963.21			
Aggregate	1	710.45	710.45	63.27	.000
Compaction	2	6,806.45	3,403.23	303.07	.000
Interaction	2	953.45	476.72	42.45	.000
Error	8	89.83	11.23		

Interpretation of the Example

The null hypothesis of no interaction between aggregate type and compaction method is

$$H_0\colon \mu_{ij} - \mu_{kj} - \mu_{im} + \mu_{km} = 0 \quad \text{for all } i, j, k, \text{ and } m \tag{6.50}$$

Interaction is significant since the statistic $F_0 = MS(AB)/MSE = 476.72/11.23 = 42.45$, in Table 6.21, is significant with $Pr > F = .000$.

With significant interaction between aggregate type and compaction method it will be necessary to look at the simple effects of one factor at each of the levels of the other factor to understand the nature of the interaction. Comparisons between the cell means for aggregate type at each level of compaction method are shown in Display 6.7.

Display 6.7 Bonferroni *t* Tests for Simple Effects of Aggregate Type for Each Compaction Method

Compaction Method	$\hat{\bar{\mu}}_{1j} - \hat{\bar{\mu}}_{2j}$	*Standard Error*	t_0
Regular	$107.0 - 111.0 = -4.0$	$\sqrt{11.23\left[\frac{1^2}{2} + \frac{(-1)^2}{3}\right]} = 3.1$	-1.29
Low	$97.3 - 61.5 = 35.8$	$\sqrt{11.23\left[\frac{1^2}{3} + \frac{(-1)^2}{2}\right]} = 3.1$	11.55
Very low	$56.0 - 41.7 = 14.3$	$\sqrt{11.23\left[\frac{1^2}{1} + \frac{(-1)^2}{3}\right]} = 3.9$	3.67

The t_0 statistics were computed for the contrast between aggregate types for each of the compaction methods in Display 6.7. The critical value for the Bonferroni t statistic with three comparisons is $|\, t_0 \,| > t_{.025,3,8} = 3.02$. There is no significant difference between tensile strengths of the basalt and silicious rock specimens with the regular kneading compaction method. The tensile strengths of the basalt specimens were significantly greater than those for the silicious rock specimens for low and very low kneading compaction methods.

Tests for Marginal Means

The tests of hypotheses for equality of the marginal means for A and B ordinarily are not considered when interaction is significant and would not be considered for the current example. However, for the sake of illustration, the procedure is illustrated for the case when no interaction exists and tests about the marginal means would be of interest. The two hypotheses to test for the asphaltic concrete example are (1) no differences among the marginal means for aggregate type

$$H_0\colon \bar{\mu}_{1.} = \bar{\mu}_{2.} \tag{6.51}$$

and (2) no differences among the marginal means for compaction method

$$H_0\colon \bar{\mu}_{.1} = \bar{\mu}_{.2} = \bar{\mu}_{.3} \tag{6.52}$$

The sum of squares for the weighted squares of means method are shown in Table 6.21. The statistic $F_0 = MSA/MSE = 710.45/11.23 = 63.26$ tests the equality of marginal means for aggregate type. The statistic for the test of equality among the marginal means of compaction method is $F_0 = MSC/MSE = 3403.23/11.23 = 303.05$. Both of the statistics are significant with $Pr > F = .000$ in Table 6.21.

Some Comments About Missing Data and Missing Cells

A method was illustrated in this section to analyze the data from a study with unequal subclass replication in a factorial treatment design. The method provides the correct estimators for population means and credible tests of hypotheses about the factors. Other methods of analysis for the unbalanced factorial treatment designs provide different tests of hypotheses about the factor effects.

Searle, Speed, and Henderson (1981) discussed five methods for calculating sums of squares in the analysis of variance that were currently used in computer programs. All of the methods give the same results for balanced data, but they can yield different results for unbalanced data. Related articles by Hocking and Speed (1975), Speed and Hocking (1976), and Speed, Hocking, and Hackney (1978) provide additional information on the computing methods and the hypotheses that are tested in the analyses by different computer programs. More extensive illustrations of some methods, including that used in this section along with some of the theoretical background, are found in Searle (1971, 1987) and Milliken and Johnson (1984).

All of the methods discussed in the context of the sum of squares partitions for main effects and interaction provide inadequate tests of hypotheses about factorial effects when entire cells of the factorial arrangement are missing. Under these circumstances an analysis based on the cell means model is recommended by Urquhart, Weeks, and Henderson (1973), Hocking and Speed (1975), Urquhart and Weeks (1978), and Searle (1987).

EXERCISES FOR CHAPTER 6

1. A chemical production process consists of a first reaction with an alcohol and a second reaction with a base. A 3 × 2 factorial experiment with three alcohols and two bases was conducted with four replicate reactions conducted in a completely randomized design. The collected data were percent yield.

	Alcohol					
Base	*1*		*2*		*3*	
1	91.3	89.9	89.3	88.1	89.5	87.6
	90.7	91.4	90.4	91.4	88.3	90.3
2	87.3	89.4	92.3	91.5	93.1	90.7
	91.5	88.3	90.6	94.7	91.5	89.8

Source: P. R. Nelson (1988), Testing for interactions using analysis of means. *Technometrics* 30, 53–61.

a. Write a linear model for this experiment, explain the terms, and compute the analysis of variance for the data.
b. Make a table of cell and marginal means, and show their respective standard errors.
c. Test the null hypotheses of no base × alcohol interaction effects. What do you conclude from the test? What do you recommend as the next step in your analysis?
d. Use multiple contrasts among cell means to help explain the interaction. For example, compare the two bases for each alcohol.
e. Conduct residual analyses with a normal plot and then with a predicted plot; also conduct a Levene (Med) test. What do you conclude?

2. A company tested two chemistry methods for the determination of serum glucose. Three pools of serum were used for the experiment. Each pool contained different levels of glucose through the addition of glucose to the base level of an existing serum pool. Three samples of serum from each pool were prepared independently for each level of glucose with each of the two chemistry methods. The concentration of glucose (mg/dl) for all samples was measured on one run of a spectrophotometer.

	Method 1			*Method 2*		
Glucose Level	*1*	*2*	*3*	*1*	*2*	*3*
	42.5	138.4	180.9	39.8	132.4	176.8
	43.3	144.4	180.5	40.3	132.4	173.6
	42.9	142.7	183.0	41.2	130.3	174.9

Source: Dr. J. Anderson, Beckman Instruments Inc.

a. Write a linear model for this experiment, explain the terms, conduct an analysis of variance for the data, and compute the residuals. Is a transformation of the data necessary? Explain.
b. If a transformation is necessary, compute the transformation for the data and the analysis of variance.
c. Test the hypotheses of no method × glucose interaction effects. What do you conclude? Should you test for main effects? Why?
d. Prepare a table of cell and marginal means with their respective standard errors.
e. Test the difference between method means for each level of glucose, and interpret the results.

3. A study of the effect of temperature on percent shrinkage in dyeing fabrics was made on two replications for each of four fabrics in a completely randomized design. The data are the percent shrinkage of two replicate fabric pieces dried at each of the four temperatures.

	Temperature			
Fabric	210° F	215° F	220° F	225°F
1	1.8, 2.1	2.0, 2.1	4.6, 5.0	7.5, 7.9
2	2.2, 2.4	4.2, 4.0	5.4, 5.6	9.8, 9.2
3	2.8, 3.2	4.4, 4.8	8.7, 8.4	13.2, 13.0
4	3.2, 3.6	3.3, 3.5	5.7, 5.8	10.9, 11.1

a. Write a linear model for the experiment, explain the terms, and compute the analysis of variance for the data.
b. Test the null hypothesis of no fabric × temperature interaction.
c. Partition the temperature main effect sum of squares into 1 degree of freedom partitions for linear and quadratic regression sum of squares, and test the null hypotheses of no linear or quadratic response to temperature.
d. Partition the temperature × fabric interaction sum of squares into temperature linear × fabric and temperature quadratic × fabric interaction sum of squares, and test the null hypotheses of no interaction for the respective partitions.
e. Prepare a profile plot of the cell means versus temperature for each fabric, and interpret the results. For example, the following questions may be asked: "How does drying temperature affect the fabric shrinkage?" "How does the relationship between shrinkage and temperature differ among the fabric types?"

4. An experiment in soil microbiology was conducted to determine the effect of nitrogen fertility on nitrogen fixation by *Rhizobium* bacteria. The experiment was conducted with four crops: alfalfa, soybeans, guar, and mungbean. Two plants were inoculated with the *Rhizobium* and grown in a Leonard jar with one of three rates of nitrogen in the media: 0, 50, or 100 ppm N. Four replications, Leonard jars, were used for each of the 12 treatment combinations. The treatments were arranged in a completely randomized design in a growth chamber. The acetylene reduction was measured for each treatment when the plants were at the flowering stage. Acetylene reduction reflects the amount of nitrogen that is fixed by the bacteria in the symbiotic relationship with the plant.

	Crop			
Nitrogen	*Alfalfa*	*Soybean*	*Guar*	*Mungbean*
0	2.6, 1.1	6.5, 2.6	0.3, 0.1	0.8, 0.9
	0.9, 1.2	3.9, 4.3	0.4, 0.4	2.2, 1.2
50	0.0, 0.0	0.6, 0.6	0.0, 0.1	0.7, 0.4
	0.0, 0.0	0.3, 0.8	0.0, 0.2	0.3, 0.8
100	0.0, 0.0	0.0, 0.1	0.0, 0.2	0.3, 0.1
	0.0, 0.0	0.1, 0.0	0.0, 0.0	0.0, 0.1

Source: Dr. I. Pepper, Department of Soil and Water Science, University of Arizona.

a. Write a linear model for this experiment, explain the terms, and compute the analysis of variance.
b. Perform a residual analysis and determine whether a transformation of the data is necessary. Transform the data if necessary, and compute the analysis of variance for the transformed data.
c. Test the null hypotheses of no crop, nitrogen, or crop $\times$ nitrogen interaction effects.
d. Partition the nitrogen main effect and the nitrogen $\times$ crop interaction sum of squares into 1 degree of freedom partitions for linear and quadratic regression.
e. Test the null hypotheses of no nitrogen linear or quadratic main effects and the null hypotheses of no nitrogen linear or quadratic interaction with crops.
f. Make a profile plot of the cell means versus level of nitrogen for each crop, and interpret the experiment. For example, you can ask the question, "How does the addition of nitrogen to the media affect the nitrogen fixation by the *Rhizobium*?" or "Is the effect of the addition of nitrogen on nitrogen fixation the same for each crop?"
g. Note that two treatment combinations, alfalfa with 50 and 100 ppm N, have all observations with a value of zero. This phenomenon is possible if the presence of a threshold level of nitrogen in the growth medium completely inhibits *Rhizobium* activity. How does this affect the assumptions for the analysis of variance regarding homogeneity of variance? Do you have any suggestions to accommodate this situation in your analysis of the data?

5. An agronomist conducted an experiment to determine the combined effects of an herbicide and an insecticide on the growth and development of cotton plants (delta pine smoothleaf). The insecticide and the herbicide were incorporated into the soil used in the containers to grow the cotton plants. Four containers each with five cotton plants were used for each treatment combination. The containers were arranged in the greenhouse in a completely randomized design. Five levels (lb/acre) were used for both the insecticide and the herbicide to give 25 treatment combinations. The data that follow are cell means for the dry weight of the roots (grams/plant) when the plants were three weeks old.

	Herbicide				
Insecticide	0	0.5	1.0	1.5	2.0
0	122.0	72.50	52.00	36.25	29.25
20	82.75	84.75	71.50	80.50	72.00
40	65.75	68.75	79.50	65.75	82.50
60	68.00	70.00	68.75	77.25	68.25
80	57.50	60.75	63.00	69.25	73.25

Mean Square for Experimental Error = 174 with 75 degrees of freedom

Source: Dr. K. Hamilton, Department of Plant Sciences, University of Arizona.

a. Compute 1 degree of freedom regression sum of squares partitions for herbicide, insecticide, and interaction sums of squares. Compute no higher polynomial than cubic regression for herbicide or insecticide.
b. Test the null hypotheses for each of the partitions, and determine the form of the polynomial regression that adequately describes the response.
c. Transform the orthogonal polynomial equation into an equation in terms of herbicide and insecticide. Either use a standard regression program or the transformation equations in Chapter 3.
d. Interpret the results from plots of the cell means or the estimated polynomial equation.

6. An experiment was conducted on the durability of coated fabric subjected to standard abrasive tests. The $2 \times 2 \times 3$ factorial design included two different fillers (F_1, F_2) in three different proportions (25%, 50%, 75%) with or without surface treatment (S_1, S_2). Two replicate fabric specimens were tested for each of the 12 treatment combinations in a completely randomized design. The data are weight loss (mg) of the fabric specimens from the abrasion test.

	Surface and Filler Treatments			
	S_1		S_2	
Proportion of Filler	F_1	F_2	F_1	F_2
25%	194	239	155	137
	208	187	173	160
50%	233	224	198	129
	241	243	177	98
75%	265	243	235	155
	269	226	229	132

Source: G. Box (1950), Problems in the analysis of growth and wear curves. Biometrics 6, 362–389.

a. Write a linear model for the experiment, explain the terms, and compute the analysis of variance for the data.
b. Prepare a table of cell and marginal means with their respective standard errors.
c. Test the null hypothesis for all main and interaction effects.
d. Compute the 1 degree of freedom regression sum of squares partitions for proportion of filler and interaction between proportion of filler and the other factors.

e. Test the null hypotheses for the regression partitions.
f. Plot cell means versus proportion of filler for the four treatment combinations of surface and filler type, and interpret the results of your analysis.

7. A soil scientist conducted an experiment to evaluate a four-electrode resistance network to compute electroconductivity (EC) of soil in specially constructed acrylic conductivity cells. The objective of the study was to evaluate the relationship between measured EC and soil water salinity at different water contents of soils. Three basic soil textures were included in the experiment since EC is specific to soil texture. The cells were constructed of acrylic tubing, 4-cm long by 8.2-cm diameter, and packed with soil. Two cells were used for each treatment combination. The three soil types were loamy sand, loam, and clay. The salinity of the soil water, three levels, was based on the EC of the water at 2, 8, and 16 dS/m (decisiemens/meter). The water content of the soil was three levels, at 0%, 5%, and 15%. The resulting experiment was a $3 \times 3 \times 3$ factorial arrangement with two replications in a completely randomized design. The EC values of the soil determined on the basis of readings from the four-electrode network follow.

Salinity	*2*			*8*			*16*		
Water	*0*	*5*	*15*	*0*	*5*	*15*	*0*	*5*	*15*
Loamy	0.60	1.69	3.47	0.05	0.11	0.06	0.07	0.08	0.22
sand	0.48	2.01	3.30	0.12	0.09	0.19	0.06	0.14	0.17
Loam	0.98	2.21	5.68	0.15	0.23	0.40	0.07	0.23	0.43
	0.93	2.48	5.11	0.26	0.35	0.75	0.21	0.35	0.35
Clay	1.37	3.31	5.74	0.72	0.78	2.10	0.40	0.72	1.95
	1.50	2.84	5.38	0.51	1.11	1.18	0.57	0.88	2.87

Source: H. Bohn and T. Tabbara, Department of Soil and Water Science, University of Arizona.

a. Write a linear model for this experiment, explain the terms, and compute the analysis of variance for the data.
b. Prepare a table of cell and marginal means and their respective standard errors.
c. Test the null hypotheses for all main effects and interactions.
d. Compute the linear and quadratic orthogonal polynomial regression sum of squares partitions for salinity and water and their interactions, including the interactions with texture. Note that the levels of salinity and water are unequally spaced; therefore, the standard orthogonal polynomial coefficients given in Appendix Table XI do not apply. Some statistical computing programs will automatically compute the orthogonal polynomial coefficients, given values for the levels of the factors (for example, MANOVA in SPSS and 2V in BMDP). The following are orthogonal polynomial coefficients that may be used to compute the orthogonal partitions:

Water linear:	−0.617	−0.154	0.772
Water quadratic:	0.535	−0.802	0.267
Salinity linear:	−0.671	−0.067	0.738
Salinity quadratic:	0.465	−0.814	0.349

8. Five nickel rods of 1-mm diameter were put in a metallic clamp in a suspension of aluminum oxide. A 100-volt tension was applied between the nickel rods and the vessel containing the suspension of aluminum oxide. The thickness of the aluminum oxide layer deposited on the nickel rods was recorded at three height positions of the five rods. The data are thickness of the deposit in microns.

	Clamp Position of Nickel Rod				
Height	*1*	*2*	*3*	*4*	*5*
1	125	130	128	134	143
2	126	150	127	124	118
3	130	155	168	159	138

Source: H. Hamaker (1955), Experimental design in industry. *Biometrics* 11, 257–286.

a. Write a linear model for this experiment, explain the terms, and state the model assumptions.
b. Do you believe the model assumptions are valid for this experiment? Explain.
c. Suppose the assumptions are reasonably valid. Compute the analysis of variance for the data.
d. Compute the 1 degree of freedom sum of squares for nonadditivity.
e. Is the additive model for clamp position and height sufficient for this data?

9. An entomologist conducted an experiment on the drinking rate energetics of honeybees to determine the effects of ambient temperature and viscosity of the liquid on energy consumption by the honeybee. The temperature levels were 20° C, 30° C, and 40° C. The viscosity of the liquid was controlled by the sucrose concentrations, which were 20, 40, and 60 percent of total dissolved solids in the drinking liquid for the bees. The entomologist recorded the energy expended by the bees as joules/second. The data given below are for three replications for each of the nine treatment combinations in a completely randomized design.

	Sucrose %		
Temperature °C	20	40	60
20	3.1, 3.7, 4.7	5.5, 6.7, 7.3	7.9, 9.2, 9.3
30	6.0, 6.9, 7.5	11.5, 12.9, 13.4	17.5, 15.8, 14.7
40	7.7, 8.3, 9.5	15.7, 14.3, 15.9	19.1, 18.0, 19.9

Source: Dr. S. Buckman, USDA Bee Research Lab, Tucson, Arizona.

a. Compute 1 degree of freedom regression sum of squares partitions for temperature, sucrose %, and interaction sums of squares.
b. Test the null hypotheses for each of the partitions, and determine the form of the polynomial regression that adequately describes the response.
c. Transform the orthogonal polynomial equation into an equation in terms of herbicide and insecticide. Either use a standard regression program or the transformation equations in Chapter 3.
d. Construct a profile plot such as that in Figure 6.5 with the estimated cell means and the estimated polynomial equation.
e. Interpret the results.

10. *Unequal Replication Numbers:* A biologist incubated adrenal glands of rats in vitro under stimulation by ACTH and measured steroid production of the glands. The glands were taken from the animals at four different stages of growth and were subjected to two different treatments. Glands from four animals were used for each treatment combination. However, several laboratory analysis were invalid, which resulted in unequal replication numbers for the treatments. The data given below are steroid production per 100 mg of gland per hour.

	Treatment	
Stage	*1*	*2*
1	6.98, 6.58	8.62, 9.40, 9.20
2	6.07, 7.16, 6.34	9.42, 6.67, 8.64
3	5.38, 7.31, 6.65, 7.44	4.96, 6.80, 7.61
4	7.02, 9.23, 7.32	7.17, 7.65, 6.52, 6.86

Source: Dr. R. Chaisson, Department of Veterinary Science, University of Arizona.

a. Compute the analysis of variance in order to test the global hypothesis of no Stage by Treatment interaction.
b. Compute the least squares means and their standard errors for marginal and cell means.
c. Estimate the contrast between the two treatments for levels 1, 2, 3, and 4 of the growth stages $(\widehat{\mu}_{1j.} - \widehat{\mu}_{2j.}$ for $j = 1, 2, 3, 4)$, and test the hypothesis of no difference at the .05 level of significance between the two means in each case.
d. How do the least squares means differ from the observed means?

11. *Unequal Replication Numbers*: Suppose the experiment on the chemical production process in Exercise 6.1 had unequal replications among the six treatment combinations of the two factors, Base and Alcohol. The data are given below.

	Alcohol		
Base	*1*	*2*	*3*
1	90.7, 91.4	89.3, 88.1 90.4	89.5, 87.6 88.3, 90.3
2	87.3, 88.3 91.5	94.7	93.1, 90.7 91.5

a. Compute the analysis of variance in order to test the global hypothesis of no Base by Alcohol interaction.
b. Compute the least squares means and their standard errors for marginal and cell means.
c. Estimate the contrast between the two bases for levels 1, 2, and 3 of Alcohol $(\widehat{\mu}_{1j.} - \widehat{\mu}_{2j.};\ j = 1, 2, 3)$, and test the hypothesis of no difference at the .05 level of significance between the two means in each case.
d. How do the least squares means differ from the observed means?

6A Appendix: Least Squares for Factorial Treatment Designs

Equal Treatment Replication

The sum of squares partitions for the data from a factorial treatment design can be derived from solutions to the least squares normal equations for a factorial effects model. The full model for a factorial treatment design with two factors will be used to illustrate the derivation.

For the sake of simplicity in notation the full model is written as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk} \qquad \textbf{(6A.1)}$$

$$i = 1, 2, \ldots, a \quad j = 1, 2, \ldots, b \quad k = 1, 2, \ldots, r$$

where μ is the general mean, α_i is the effect of factor A, β_j is the effect of factor B, γ_{ij} is the interaction effect, and e_{ijk} is the random independent experimental error. The interaction term $(\alpha\beta)_{ij}$ used in the main body of this chapter has been replaced by γ_{ij} to simplify the notation for the presentation in this appendix.

The least squares estimates for the parameters in the full model are those that minimize the sum of squares for experimental error

$$Q = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r} e_{ijk}^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{r} (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 \qquad \textbf{(6A.2)}$$

The normal equations from the minimization include one equation for μ and one equation for each of the factorial effects, $\alpha_1, \alpha_2, \ldots, \alpha_a$; $\beta_1, \beta_2, \ldots, \beta_b$; and $\gamma_{11}, \gamma_{12}, \ldots, \gamma_{ab}$. The normal equations are obtained from the following set of derivatives:

$$\begin{aligned}
&\frac{\partial}{\partial \mu} \sum_i\sum_j\sum_k (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 = 0 \\
&\frac{\partial}{\partial \alpha_i} \sum_j\sum_k (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 = 0 \quad i = 1, 2, \ldots, a \\
&\frac{\partial}{\partial \beta_j} \sum_i\sum_k (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 = 0 \quad j = 2, 3, \ldots, b \\
&\frac{\partial}{\partial \gamma_{ij}} \sum_k (y_{ijk} - \mu - \alpha_i - \beta_j - \gamma_{ij})^2 = 0 \quad i = 1, 2, \ldots, a \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad j = 1, 2, \ldots, b
\end{aligned} \qquad \textbf{(6A.3)}$$

Simplifying, the set of normal equations to solve is

$$\begin{aligned}
\mu:&\quad abr\widehat{\mu} + br\sum_j \widehat{\alpha}_i + ar\sum_j \widehat{\beta}_j + r\sum_i\sum_j \widehat{\gamma}_{ij} = y_{...} \\
\alpha_i:&\quad br\widehat{\mu} + br\widehat{\alpha}_i + r\sum_j \widehat{\beta}_j + r\sum_j \widehat{\gamma}_{ij} = y_{i..} \quad i = 1,2,\ldots,a \\
\beta_j:&\quad ar\widehat{\mu} + r\sum_i \widehat{\alpha}_i + ar\widehat{\beta}_j + r\sum_i \widehat{\gamma}_{ij} = y_{.j.} \quad j = 1,2,\ldots,b \\
\gamma_{ij}:&\quad r\widehat{\mu} + r\widehat{\alpha}_i + r\widehat{\beta}_j + r\widehat{\gamma}_{ij} = y_{ij.} \quad i = 1,2,\ldots,a;\ j = 1,2,\ldots,b
\end{aligned} \tag{6A.4}$$

Upon close inspection the a equations derived for the factor A effects sum to the first equation for μ; the b equations derived for the factor B effects sum to the first equation for μ as do the ab equations for interaction; the γ_{ij} equations summed over the j subscript will give the α_i equation; and the γ_{ij} equations summed over the i subscript will give the β_j equation. These linear dependencies require constraints imposed on the estimates to provide a unique solution to the equations. Any constraints that lead to a solution will suffice. One set of constraints commonly used are the sum-to-zero constraints. The sum-to-zero constraints are $\Sigma\widehat{\alpha}_i = 0$, $\Sigma\widehat{\beta}_j = 0$, $\Sigma_i^a\widehat{\gamma}_{ij} = 0\ (j = 1,2,\ldots,b)$, and $\Sigma_j^b\widehat{\gamma}_{ij} = 0\ (i = 1,2,\ldots,a)$.

With the constraints, the equations are

$$\begin{aligned}
\mu:&\quad abr\widehat{\mu} = y_{...} \\
\alpha_i:&\quad br\widehat{\mu} + br\widehat{\alpha}_i = y_{i..} \quad i = 1,2,\ldots,a \\
\beta_j:&\quad ar\widehat{\mu} + ar\widehat{\beta}_j = y_{.j.} \quad j = 1,2,\ldots,b \\
\gamma_{ij}:&\quad r\widehat{\mu} + r\widehat{\alpha}_i + r\widehat{\beta}_j + r\widehat{\gamma}_{ij} = y_{ij.} \quad i = 1,2,\ldots,a;\ j = 1,2,\ldots,b
\end{aligned} \tag{6A.5}$$

The solutions are

$$\begin{aligned}
\widehat{\mu} &= \frac{y_{...}}{abr} = \overline{y}_{...} \\
\widehat{\alpha}_i &= \frac{y_{i..}}{br} - \widehat{\mu} = \overline{y}_{i..} - \overline{y}_{...} \quad i = 1,2,\ldots,a \\
\widehat{\beta}_j &= \frac{y_{.j.}}{ar} - \widehat{\mu} = \overline{y}_{.j.} - \overline{y}_{...} \quad j = 1,2,\ldots,b \\
\widehat{\gamma}_{ij} &= \frac{y_{ij.}}{r} - \widehat{\mu} - \widehat{\alpha}_i - \widehat{\beta}_j = \overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...} \quad i = 1,2,\ldots,a;\ j = 1,2,\ldots,b
\end{aligned} \tag{6A.6}$$

The estimate of the sum of squares for experimental error is obtained with a substitution of the estimates $\widehat{\mu}$, $\widehat{\alpha}_i$, $\widehat{\beta}_j$, and $\widehat{\gamma}_{ij}$ into Equation (6A.2) as

$$SSE_f = \sum_i\sum_j\sum_k (y_{ijk} - \widehat{\mu} - \widehat{\alpha}_i - \widehat{\beta}_j - \widehat{\gamma}_{ij})^2 = \sum_i\sum_j\sum_k (y_{ijk} - \overline{y}_{ij.})^2 \tag{6A.7}$$

The difference between the total sum of squares and SSE_f is known as the reduction in sum of squares due to fitting the model and is sometimes written as $R(\mu, \alpha, \beta, \gamma)$. With equal replication for all treatment combinations the sum of squares for each factorial effect can be derived from the computation used to compute $R(\mu, \alpha, \beta, \gamma)$. The reduction in sum of squares due to fitting the full model is

$$R(\mu, \alpha, \beta, \gamma) = \widehat{\mu}y_{...} + \sum_{i=1}^{a}\widehat{\alpha}_i y_{i..} + \sum_{j=1}^{b}\widehat{\beta}_j y_{.j.} + \sum_{i=1}^{a}\sum_{j=1}^{b}\widehat{\gamma}_{ij}y_{ij.} \tag{6A.8}$$

For balanced data with equal replication numbers for each treatment combination, the sum of squares partitions for the analysis of variance can be taken from the individual terms in Equation (6A.8) as

$$\begin{aligned}
CF &= \widehat{\mu}y_{...} = \frac{(y_{...})^2}{abr} \\
SSA &= \sum_{i=1}^{a}\widehat{\alpha}_i y_{i..} = br\sum_{i=1}^{a}(\overline{y}_{i..} - \overline{y}_{...})^2 \\
SSB &= \sum_{j=1}^{b}\widehat{\beta}_j y_{.j.} = ar\sum_{j=1}^{b}(\overline{y}_{.j.} - \overline{y}_{...})^2 \\
SS(AB) &= \sum_{i=1}^{a}\sum_{j=1}^{b}\widehat{\gamma}_{ij}y_{ij.} = r\sum_{i=1}^{a}\sum_{j=1}^{b}(\overline{y}_{ij.} - \overline{y}_{i..} - \overline{y}_{.j.} + \overline{y}_{...})^2
\end{aligned} \tag{6A.9}$$

The sum of squares partitions shown in Equation (6A.9) are those shown in Section 6.4. They may be derived from considerations of full and reduced models. For example, the sum of squares for interaction $SS(AB)$ can be found as the difference between the experimental error sums of squares for the reduced model without the interaction terms and full model with interaction terms included. The models and sums of squares are

Full Model: $y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$ with SSE_f

Reduced Model: $y_{ijk} = \mu + \alpha_i + \beta_j + e_{ijk}$ with SSE_r

The interaction sum of squares is found as $SS(AB) = SSE_r - SSE_f$, which is $\Sigma_i\Sigma_j\widehat{\gamma}_{ij}y_{ij.}$ [the same as the last term shown in Equation (6A.8)].

The equivalence of $SSE_r - SSE_f$ to $\Sigma_i\Sigma_j\widehat{\gamma}_{ij}y_{ij.}$ can be shown by solving the normal equations for the reduced model and computing the reduction in sum of squares due to fitting the reduced model as $R(\mu, \alpha, \beta)$. The normal equations for the reduced model are obtained from those for the full model in Equation (6A.5) by removing the equations for γ_{ij} and the $\widehat{\gamma}_{ij}$ terms in the remaining equations. The solutions for $\widehat{\mu}$, $\widehat{\alpha}_i$, and $\widehat{\beta}_j$ will be those shown in Equation (6A.6). The reduction in sum of squares due to fitting the reduced model will be

$$R(\mu,\alpha,\beta) = \widehat{\mu}y_{...} + \sum_{i=1}^{a}\widehat{\alpha}_i y_{i..} + \sum_{j=1}^{b}\widehat{\beta}_j y_{.j.} \qquad \textbf{(6A.10)}$$

The difference between $R(\mu,\alpha,\beta,\gamma)$ and $R(\mu,\alpha,\beta)$ is seen to be $\Sigma_i\Sigma_j\widehat{\gamma}_{ij}y_{ij.}$, and, therefore, the differences between the sums of squares for experimental error of the two models will be equivalent to the same quantity. That is,

$$SS(\boldsymbol{AB}) = SSE_r - SSE_f = R(\mu,\alpha,\beta,\gamma) - R(\mu,\alpha,\beta) \qquad \textbf{(6A.11)}$$

The sums of squares $SS\boldsymbol{A}$ and $SS\boldsymbol{B}$ can be derived in a similar fashion for balanced data, or $SS\boldsymbol{A} = R(\mu,\alpha) - R(\mu)$ and $SS\boldsymbol{B} = R(\mu,\beta) - R(\mu)$.

Unequal Treatment Replication

The derivation of the interaction sum of squares with unequal treatment replication in the factorial treatment design is demonstrated with the simple 2×2 factorial example shown in Table 6A.1.

Table 6A.1 Example data for a 2×2 factorial with unequal treatment replication

		B		
		1	*2*	$y_{i..}$
A	1	6,5,3 $r_{11} = 3$	2,4 $r_{12} = 2$	20 $r_{1.} = 5$
	2	5,4 $r_{21} = 2$	3 $r_{22} = 1$	12 $r_{2.} = 3$
	$y_{.j.}$	23 $r_{.1} = 5$	9 $r_{.2} = 3$	32 $r_{..} = 8$

Full Model

The full model normal equations for the example data in Table 6A.1 are derived by the methods shown at the beginning of this appendix. The coefficients for the parameters in the equations will reflect the unequal replication numbers. In general, the equations will be

$$\mu: \quad N\widehat{\mu} + \sum_{i=1}^{a} r_{i.}\widehat{\alpha}_i + \sum_{j=1}^{b} r_{.j}\widehat{\beta}_j + \sum_{r=1}^{a}\sum_{j=1}^{b} r_{ij}\widehat{\gamma}_{ij} = y_{...}$$

$$\alpha_i: \quad r_{i.}\widehat{\mu} + r_{i.}\widehat{\alpha}_i + \sum_{j=1}^{b} r_{ij}\widehat{\beta}_j + \sum_{j=1}^{b} r_{ij}\widehat{\gamma}_{ij} = y_{i..}$$

$$\beta_j: \quad r_{.j}\widehat{\mu} + \sum_{i=1}^{a} r_{ij}\widehat{\alpha}_i + r_{.j}\widehat{\beta}_j + \sum_{i=1}^{a} r_{ij}\widehat{\gamma}_{ij} = y_{.j.}$$

$$\gamma_{ij}: \quad r_{ij}\widehat{\mu} + r_{ij}\widehat{\alpha}_i + r_{ij}\widehat{\beta}_j + r_{ij}\widehat{\gamma}_{ij} = y_{i..}$$

The normal equations for the data in Table 6A.1 are

$$\begin{array}{rcrcrcrcrcrcrcrcrcl}
8\widehat{\mu} & + & 5\widehat{\alpha}_1 & + & 3\widehat{\alpha}_2 & + & 5\widehat{\beta}_1 & + & 3\widehat{\beta}_2 & + & 3\widehat{\gamma}_{11} & + & 2\widehat{\gamma}_{12} & + & 2\widehat{\gamma}_{21} & + & \widehat{\gamma}_{22} & = & 32 \\
5\widehat{\mu} & + & 5\widehat{\alpha}_1 & & & + & 3\widehat{\beta}_1 & + & 2\widehat{\beta}_2 & + & 3\widehat{\gamma}_{11} & + & 2\widehat{\gamma}_{12} & & & & & = & 20 \\
3\widehat{\mu} & & & + & 3\widehat{\alpha}_2 & + & 2\widehat{\beta}_1 & + & \widehat{\beta}_2 & + & & & & + & 2\widehat{\gamma}_{21} & + & \widehat{\gamma}_{22} & = & 12 \\
5\widehat{\mu} & + & 3\widehat{\alpha}_1 & + & 2\widehat{\alpha}_2 & + & 5\widehat{\beta}_1 & & & + & 3\widehat{\gamma}_{11} & & & + & 2\widehat{\gamma}_{21} & & & = & 23 \\
3\widehat{\mu} & + & 2\widehat{\alpha}_1 & + & \widehat{\alpha}_2 & & & + & 3\widehat{\beta}_2 & & & + & 2\widehat{\gamma}_{12} & & & + & \widehat{\gamma}_{22} & = & 9 \\
3\widehat{\mu} & + & 3\widehat{\alpha}_1 & & & + & 3\widehat{\beta}_1 & & & + & 3\widehat{\gamma}_{11} & & & & & & & = & 14 \\
2\widehat{\mu} & + & 2\widehat{\alpha}_1 & & & & & + & 2\widehat{\beta}_2 & & & + & 2\widehat{\gamma}_{12} & & & & & = & 6 \\
2\widehat{\mu} & & & + & 2\widehat{\alpha}_2 & + & 2\widehat{\beta}_1 & & & & & & & + & 2\widehat{\gamma}_{21} & & & = & 9 \\
\widehat{\mu} & & & + & \widehat{\alpha}_2 & & & + & \widehat{\beta}_2 & & & & & & & + & \widehat{\gamma}_{22} & = & 3
\end{array}$$

Since there are linear dependencies in the equations the "sum-to-zero constraints" can be imposed on the equations to obtain a solution. The constraints are

$$\widehat{\alpha}_1 + \widehat{\alpha}_2 = 0,\ \widehat{\beta}_1 + \widehat{\beta}_2 = 0,\ \widehat{\gamma}_{11} + \widehat{\gamma}_{11} + \widehat{\gamma}_{12} = 0,\ \widehat{\gamma}_{11} + \widehat{\gamma}_{21} = 0$$

$$\widehat{\gamma}_{21} + \widehat{\gamma}_{21} = 0, \text{ and } \widehat{\gamma}_{12} + \widehat{\gamma}_{22} = 0$$

The solutions to the equations after the constraints are applied are

$$\widehat{\mu} = \frac{91}{24},\ \widehat{\alpha}_1 = \frac{1}{24},\ \widehat{\alpha}_2 = -\frac{1}{24},\ \widehat{\beta}_1 = \frac{19}{24},\ \widehat{\beta}_2 = -\frac{19}{24}$$

$$\widehat{\gamma}_{11} = \frac{1}{24},\ \widehat{\gamma}_{12} = -\frac{1}{24},\ \widehat{\gamma}_{21} = -\frac{1}{24},\ \widehat{\gamma}_{22} = \frac{1}{24}$$

The sum of squares for experimental error can be determined with

$$SSE_f = \sum_i\sum_j\sum_k y_{ijk}^2 - R(\mu,\alpha,\beta,\gamma)$$

where

$$R(\mu,\alpha,\beta,\gamma) = \widehat{\mu}y_{...} + \sum_i \widehat{\alpha}_i y_{i..} + \sum_j \widehat{\beta}_j y_{.j.} + \sum_i\sum_j \widehat{\gamma}_{ij} y_{ij.}$$

The calculation for $R(\mu,\alpha,\beta,\gamma)$ is

$$\begin{aligned} R(\mu,\alpha,\beta,\gamma) &= \widehat{\mu}y_{...} + \widehat{\alpha}_1 y_{1..} + \widehat{\alpha}_2 y_{2..} + \widehat{\beta}_1 y_{.1.} + \widehat{\beta}_2 y_{.2.} \\ &\quad + \widehat{\gamma}_{11} y_{11.} + \widehat{\gamma}_{12} y_{12.} + \widehat{\gamma}_{21} y_{21.} + \widehat{\gamma}_{22} y_{22.} \\ &= \frac{1}{24}[91(32) + 1(20) + (-1)(12) + 19(23) + \cdots + (-1)(9) + 1(3)] \\ &= 132.833 \end{aligned}$$

Given $\Sigma y_{ijk}^2 = 140$, the sum of squares for experimental error from the full model is

$$SSE_f = 140 - 132.833 = 7.167$$

Reduced Model

The normal equations for the reduced model without the γ_{ij} interaction terms are obtained by eliminating the equations for the γ_{ij} and eliminating the $\widehat{\gamma}_{ij}$ terms from the remaining equations shown for the full model. The equations for the reduced model are

$$\begin{aligned} 8\widehat{\mu} + 5\widehat{\alpha}_1 + 3\widehat{\alpha}_2 + 5\widehat{\beta}_1 + 3\widehat{\beta}_2 &= 32 \\ 5\widehat{\mu} + 5\widehat{\alpha}_1 + 3\widehat{\beta}_1 + 2\widehat{\beta}_2 &= 20 \\ 3\widehat{\mu} + 3\widehat{\alpha}_2 + 2\widehat{\beta}_1 + \widehat{\beta}_2 &= 12 \\ 5\widehat{\mu} + 3\widehat{\alpha}_1 + 2\widehat{\alpha}_2 + 5\widehat{\beta}_1 &= 23 \\ 3\widehat{\mu} + 2\widehat{\alpha}_1 + \widehat{\alpha}_2 + 3\widehat{\beta}_2 &= 9 \end{aligned}$$

The sum-to-zero constraints are $\widehat{\alpha}_1 + \widehat{\alpha}_2 = 0$ and $\widehat{\beta}_1 + \widehat{\beta}_2 = 0$. The solutions are

$$\widehat{\mu} = \frac{212}{56},\ \widehat{\alpha}_1 = \frac{3}{56},\ \widehat{\alpha}_2 = -\frac{3}{56},\ \widehat{\beta}_1 = \frac{45}{56},\ \widehat{\beta}_2 = -\frac{45}{56}$$

The reduction in sum of squares due to fitting the reduced model is

$$\begin{aligned} R(\mu,\alpha,\beta) &= \widehat{\mu}y_{...} + \widehat{\alpha}_1 y_{1..} + \widehat{\alpha}_2 y_{2..} + \widehat{\beta}_1 y_{.1.} + \widehat{\beta}_2 y_{.2.} \\ &= \frac{1}{56}[212(32) + 3(20) + (-3)(12) + 45(23) + (-45)(9)] \\ &= 132.821 \end{aligned}$$

The sum of squares for experimental error for the reduced model is

$$SSE_r = \sum_i\sum_j\sum_k y_{ijk}^2 - R(\mu,\alpha,\beta) = 140 - 132.821 = 7.179$$

The sum of squares for interaction is

$$SS(\boldsymbol{AB}) = R(\mu,\alpha,\beta,\gamma) - R(\mu,\alpha,\beta) = 132.833 - 132.821 = 0.012$$

$$SS(\boldsymbol{AB}) = SSE_r - SSE_f = 7.179 - 7.167 = 0.012$$

Main Effect Sums of Squares

Tests of equality for marginal means for A and B, H_0: $\overline{\mu}_{1.} = \overline{\mu}_{2.}$ and H_0: $\overline{\mu}_{.1} = \overline{\mu}_{.2}$, in the absence of interaction require the sums of squares partitions from the method of weighted squares of means (Yates, 1934). Some computer programs that compute the required sums of squares were indicated in Section 6.9. The hypothesis tested by the sum of squares partition for a main effect depends greatly on the computational technique used in the least squares estimation process. Details of the results from different techniques can be found in Hocking and Speed (1975), Speed and Hocking (1976), Speed, Hocking, and Hackney (1978), and Searle, Speed, and Henderson (1981).