5 Experiments to Study Variances

In this chapter, the statistical model for research studies about the variances of populations is introduced. Knowledge about the assignable causes of variation is useful for improving manufacturing processes, improving the genetics of crops and livestock, enhancing quality control in the health industry, and designing research studies. The objective is to decompose the total variance into identifiable components.

5.1 Random Effects Models for Variances

The meat storage experiment from Chapters 2 and 3 included four specific treatments with no expressed interest in any other packagings for the experiment. Thus, the complete treatment population of interest consisted of the four packaging methods.

Each of the four packaging methods could be duplicated if the experiment was repeated. Under these circumstances, the statistical models used for the studies are referred to as **fixed effects** models, and the inferences are restricted to the particular set of treatments in the study.

There are other types of research studies in which we want to identify the major sources of variability in a system and estimate their variances. By nature of (1) the research objectives, (2) the treatment structure, (3) the experimental protocols, and (4) the type of inferences made from the observed results, the effects in the model are considered to be random effects, and the statistical models are referred to as **random effects** models. The following example illustrates a system in which knowledge about variability in identifiable components of an industrial process can be used to improve the process product.

Example 5.1 Castings of High Temperature Alloys

A metal alloy is produced in a high-temperature casting process. Each casting is broken down into smaller individual bars that are used in applications requiring small amounts of the alloy. The tensile strength of the alloy is critical to its intended future use.

The casting process is designed to produce bars with an average tensile strength above minimum specifications. Some variation in tensile strength among the bars is acceptable when only a small proportion of bars do not meet specifications (Figure 5.1(a)). However, excessive variation results in an unacceptable proportion of bars that do not meet specifications (Figure 5.1(b)).

Figure 5.1 Acceptable (a) and unacceptable (b) variation in tensile strength

Two components contribute to the total variation in tensile strength of the manufactured bars: variability among fabrication castings and inconsistencies within the casting process that affect bars from the same casting. Maintaining control over the variation requires knowledge of the variability contributions by each part of the process.

An experiment was planned to isolate the variation in tensile strength due to the effects of different castings from that attributable to inconsistencies within the same casting.

High-temperature castings of the alloy were taken from three randomly selected fabrications conducted in the same facility. Each casting was broken down into individual bars. Destructive tensile strength measurements were obtained on a random sample of 10 bars from each of three castings. The tensile strength data for each of the 30 bars in pounds per square inch (psi) are given in Table 5.1.

The three castings used in the study represent a sample of the potential population of castings that could be produced in the facility. The investigators were interested in the variation in tensile strength among castings produced by the facility; thus, the concern was not with the three specific castings in the experiment.

The investigators considered the castings only a random sample of three from a population of castings produced by the facility. The effects of the castings will be random effects since they are randomly selected from a potentially infinite population of castings. The inferences will extend to the population of castings that

Table 5.1 Tensile strengths (psi) of bars from three separate castings of a high-temperature alloy

Casting		
1	*2*	*3*
88.0	85.9	94.2
88.0	88.6	91.5
94.8	90.0	92.0
90.0	87.1	96.5
93.0	85.6	95.6
89.0	86.0	93.8
86.0	91.0	92.5
92.9	89.6	93.2
89.0	93.0	96.2
93.0	87.5	92.5

Source: G. J. Hahn and T. E. Raghunathan (1988), Combining information from various sources: A prediction problem and other industrial applications. *Technometrics* 30, 41–52.

conceivably could be produced in the facility. Likewise, the individual bars are a random sample of bars possible from a single casting, and their effects on tensile strength are random effects.

The observed tensile strength of a particular bar (y) differs from the mean of the process (μ) by some overall error, $\delta = y - \mu$. The components of the overall error are illustrated in Figure 5.2. The overall error is the sum of two components, $\delta = \delta_c + \delta_b$, where δ_c is the error component for castings and δ_b is the error component for bars in a casting.

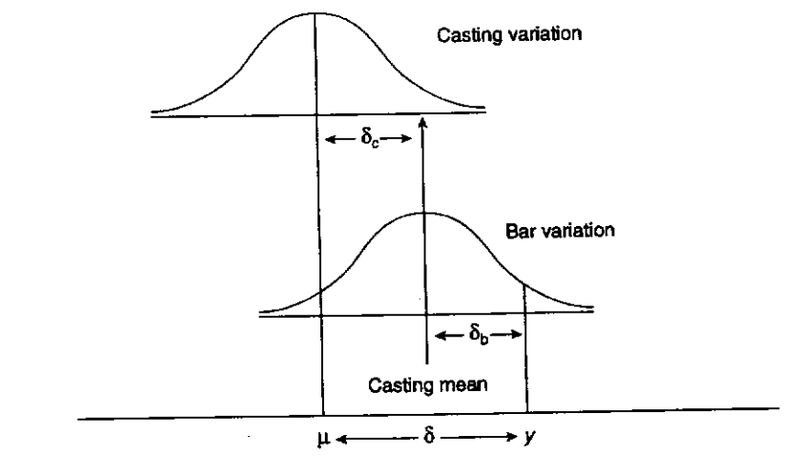

Figure 5.2 Two components of error in the metal casting process

Other Examples: Another typical study with random effects involves the inheritance of quantitative traits, such as grain yield in cultivated plant species. Many genetically distinct families of the crop are developed in a plant-breeding program. The families represent a random sample of potential families that can be developed by the plant breeder. The progeny of each family are regarded as a random sample of the progeny possible from the family. The plant breeder wants to partition the total variation into the separate contributions from the families and the progeny.

Clinical medical laboratories routinely participate in interlaboratory studies on the variability of assay results requiring random effects models for the statistical analysis. At regular intervals samples from a large homogeneous pool of serum are sent to a large number of laboratories for analysis. The participating laboratories and the samples sent to them represent a random sample of the potential populations of laboratories and serum samples. The investigators want to know if there is significant variation in assay results among the laboratories.

5.2 A Statistical Model for Variance Components

A suitable model to identify the sources of variation for the random effects in the experiment on casting high-temperature alloys is

$$y_{ij} = \mu + a_i + e_{ij} \qquad (5.1)$$

$$i = 1, 2, \ldots, t \qquad j = 1, 2, \ldots, r$$

where μ is the process mean, the a_i are the random casting effects, and the e_{ij} are the random error effects for bars within castings. The effects e_{ij} and a_i are assumed to be independent of one another.

The e_{ij} error effects are assumed to be a random sample from a population with a mean of 0 and variance σ_e^2. The random effects for the groups (a_i) are assumed to be a random sample from a population with a mean of 0 and variance σ_a^2. If $\sigma_a^2 = 0$, then all group effects are equal, but if $\sigma_a^2 > 0$ there is variability among the group effects. Since the group effects in the experiment are only a sample from a larger population of effects, the differences among the specific group means, $\mu + a_i$, are of no particular interest. The variance of the distribution of group effects, σ_a^2, is the focus of interest with random effects.

The variance of an observation, σ_y^2, may be expressed as the sum of the two variances, or $\sigma_y^2 = \sigma_a^2 + \sigma_e^2$. The variances σ_a^2 and σ_e^2 are called *components of variance,* and the model in Equation (5.1) often is referred to as a **variance components** model. In the plant-breeding study the variance component among groups (σ_a^2) represents genetic variation among families, and the plant-breeder may be interested in the ratio of this genetic variation to the total variation (σ_y^2). The engineer may use the estimate of σ_y^2 to compute percentile values for the

distribution of tensile strengths of the bars when they are to be used in a critical application.

An outline of the analysis of variance for the observations is shown in Table 5.2 with the expected mean squares for the Among Groups and Within Groups mean squares. The terms *Among Groups* and *Within Groups* will be used in place of *Treatments* and *Error* for the sources of variation to distinguish the random effects model from the fixed effects model. The computations for the sums of squares for Among Groups and Within Groups are the same as those for the Treatment and Error sums of squares given in Chapter 2 for the fixed effects model.

Table 5.2 Analysis of variance for the one-way classification with expected mean squares for the random effects model

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*Expected Mean Square*
Total	$N-1$	*SS* Total		
Among Groups	$t-1$	*SSA*	*MSA*	$\sigma_e^2 + r\sigma_a^2$
Within Groups	$N-t$	*SSW*	*MSW*	σ_e^2

5.3 Point Estimates of Variance Components

The analysis of variance method is used to estimate the variance components. The analysis of variance is computed as if the model is a fixed effects model, and the expected mean squares are derived under the assumption of the random effects model (Table 5.2). The observed mean squares are estimates of the expected mean squares, or

$$MSA = \hat{\sigma}_e^2 + r\hat{\sigma}_a^2$$

and

$$MSW = \hat{\sigma}_e^2 \tag{5.2}$$

The analysis of variance estimators of the variance components are determined by solving Equations (5.2) for the two unknowns. The solutions are

$$\hat{\sigma}_e^2 = MSW$$

and

$$\hat{\sigma}_a^2 = \frac{(MSA - MSW)}{r} \tag{5.3}$$

The estimators in Equations (5.3) are unbiased and they have the smallest variance of all estimators, which are both quadratic functions of the observations and unbiased estimators of σ_e^2 and σ_a^2.

The random effects of the model are assumed to have a normal distribution. Given the assumption of normally distributed effects, the significance of the Among Groups component of variance can be tested. The null and alternate hypotheses are H_0: $\sigma_a^2 = 0$ and H_a: $\sigma_a^2 > 0$, respectively. The test statistic is $F_0 = MSA/MSW$, and the null hypothesis is rejected at the α level of significance if $F_0 > F_{\alpha,(t-1),(N-t)}$.

The analysis of variance for the tensile strength data in Table 5.1 is shown in Table 5.3. The estimate of the components of variance for bars within castings is the mean square for Within Groups, or

$$\hat{\sigma}_e^2 = MSW = 5.82$$

The estimate of the Among Groups component of variance is

$$\hat{\sigma}_a^2 = \frac{(MSA - MSW)}{r} = \frac{(73.94 - 5.82)}{10} = 6.81$$

The estimated total variance of an observation on tensile strength is $\hat{\sigma}_y^2 = \hat{\sigma}_a^2 + \hat{\sigma}_e^2 = 6.81 + 5.82 = 12.63$.

The F_0 ratio to test the null hypothesis H_0: $\sigma_a^2 = 0$ is $F_0 = MSA/MSW = 73.94/5.82 = 12.71$. The null hypothesis is rejected with a probability of exceeding $F_0 = 12.71$ equal to .000 (Table 5.3). The castings variation contributes significantly to the variation in the tensile strengths of the alloy.

Table 5.3 Analysis of variance for tensile strengths of bars from three castings of a high temperature alloy

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Squares*	*F*	*Pr > F*
Total	29	304.99			
Among Groups	2	147.88	73.94	12.71	.000
Within Groups	27	157.10	5.82		

5.4 Interval Estimates for Variance Components

Confidence interval estimates can be computed for both variance components. The exact $100(1-\alpha)\%$ confidence interval estimator for σ_e^2 is

$$\frac{SSW}{A} < \sigma_e^2 < \frac{SSW}{B} \tag{5.4}$$

where $A = \chi^2_{\alpha/2,(N-t)}$ and $B = \chi^2_{(1-\alpha/2),(N-t)}$. A and B are values of the chi-square variable exceeded with probabilities $\alpha/2$ and $(1-\alpha/2)$, respectively. Values of chi-square are found in Appendix Table III.

An interval with at least $100(1-2\alpha)\%$ confidence for σ_a^2 is

$$\frac{SSA(1-F_u/F_0)}{rC} < \sigma_a^2 < \frac{SSA(1-F_l/F_0)}{rD} \tag{5.5}$$

where $C = \chi^2_{\alpha/2,(t-1)}$, $D = \chi^2_{(1-\alpha/2),(t-1)}$, and $F_0 = MSA/MSW$ is the observed F_0 statistic (Williams, 1962). The quantities $F_u = F_{\alpha/2,(t-1),(N-t)}$ and $F_l = F_{(1-\alpha/2),(t-1),(N-t)}$ are values of the F variable exceeded with probabilities $\alpha/2$ and $(1-\alpha/2)$, respectively.[1]

Given $SSW = 157.10$, $\chi^2_{.05,27} = 40.1$, and $\chi^2_{.95,27} = 16.2$, the 90% confidence interval estimate of σ_e^2 from Equation (5.4) is

$$\frac{157.10}{40.1} < \sigma_e^2 < \frac{157.10}{16.2}$$

$$3.92 < \sigma_e^2 < 9.70$$

With $SSA = 147.88$, $r = 10$, $\chi^2_{.025,2} = 7.38$, $\chi^2_{.975,2} = 0.05$, $F_0 = 12.71$, $F_{.025,2,27} = 4.24$, and $F_{.975,2,27} = 0.025$, the 90% confidence interval estimate of σ_a^2 from Equation (5.5) is

$$147.88\left[\frac{1-\frac{4.24}{12.71}}{10(7.38)}\right] < \sigma_a^2 < 147.88\left[\frac{1-\frac{0.025}{12.71}}{10(0.05)}\right]$$

$$1.34 < \sigma_a^2 < 295.18$$

The interval estimates for variance components will be quite wide when mean squares have small degrees of freedom. More groups of castings would provide a more precise interval estimate of σ_a^2.

Interpretations of the Variance Components

The mean tensile strength for the experiment was $\bar{y}_{..} = 90.9$ with standard error estimate

$$s_{\bar{y}_{..}} = \sqrt{\frac{MSA}{rt}} = \sqrt{\frac{73.94}{30}} = 1.57 \text{ psi}$$

The estimated variance of an observation on tensile strength on bars is $\hat{\sigma}_y^2 = \hat{\sigma}_a^2 + \hat{\sigma}_e^2 = 12.63$ with a standard deviation of $\hat{\sigma}_y = \sqrt{12.63} = 3.55$ psi. The variance component estimates isolated the different sources of variation in the casting process for alloy bars: The variance among castings accounted for 54% of the variation, and the variance among bars within castings accounted for 46% of

[1] The value of $F_{(1-\alpha)}$ cannot be read directly from Appendix Table IV, but its value can be determined from the relationship $F_{\nu_1,\nu_2,(1-\alpha)} = 1/F_{\nu_2,\nu_1,\alpha}$.

the variation. The engineer can reduce the standard deviation of $\hat{\sigma}_y = 3.55$ psi by identifying and adjusting factors in the casting process that increase variation. The variation among castings can be caused by inconsistent alloy mixtures or temperature settings from casting to casting. The variation among bars within castings can be caused by inconsistent cooling conditions or variations in the tensile strength measurement procedure.

5.5 Courses of Action with Negative Variance Estimates

By definition, a variance component is positive. However, estimates of σ_a^2 using Equation (5.3) may be negative. There are several suggested courses of action in the case of negative estimates (Searle, 1971); Searle, Casella, & McCulloch (1992).

1. Accept the estimate as evidence of a true value of zero and use zero as the estimate, recognizing that the estimator will no longer be unbiased.
2. Retain the negative estimate, recognizing that subsequent calculations using the results may not make much sense.
3. Interpret the negative component estimate as indication of an incorrect statistical model.
4. Utilize a method different from the analysis of variance for estimating the variance components.
5. Collect more data and analyze them separately or in conjunction with the existing data and hope that increased information will yield positive estimates.

Searle (1971, Chapter 9) and Searle et al. (1992, Chapter 4) discuss several methods of estimation from the extensive literature on variance component estimation, as well as these other actions, in greater detail.

5.6 Intraclass Correlation Measures Similarity in a Group

The intraclass correlation coefficient is a measure of the similarity of observations within groups relative to that among groups. When the similarity of the observations within groups is very high, σ_e^2 will be very small. Consequently, σ_a^2 will be a larger proportion of the total variation ($\sigma_y^2 = \sigma_a^2 + \sigma_e^2$). The intraclass correlation, defined as the ratio

$$\rho_I = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_e^2} \tag{5.6}$$

is used in various disciplines. Applications arise in genetics studies with various measures for the heritability of quantitative traits, in reliability studies to measure the similarity of products from the same machine or process, in medical studies to measure the repeatability of successive measurements on patients, and in survey sampling to measure the similarity of responses among people contacted by the same interviewer (Koch, 1983).

The analysis of variance was introduced by R. A. Fisher in the 1920s with an intraclass correlation model (Fisher, 1960). The model assumes all observations (y_{ij}) have the same mean (μ) and variance (σ^2), and any two members of the same group have a common correlation (ρ_I). With this model the expected mean squares for the analysis of variance in Table 5.2 are

$$E(MSA) = \sigma^2 \{1 + (r-1)\rho_I\}$$

and **(5.7)**

$$E(MSW) = \sigma^2(1 - \rho_I)$$

The estimators of ρ_I and σ^2 are found by equating the observed mean squares to the expectations shown in Equation (5.7) and solving for the unknowns. The solutions are

$$\widehat{\sigma}^2 = \frac{\{MSA + (r-1)MSW\}}{r}$$

and **(5.8)**

$$\widehat{\rho}_I = \frac{(MSA - MSW)}{\{MSA + (r-1)MSW\}}$$

The estimate of the intraclass correlation can have a minimum value of $-1/(r-1)$ and a maximum value of 1 (Fisher, 1960) because the expected value of MSA must be equal to or greater than zero.

The $100(1-\alpha)\%$ confidence interval estimator for ρ_I is

$$\frac{F_0 - F_u}{F_0 + (r-1)F_u} < \rho_I < \frac{F_0 - F_l}{F_0 + (r-1)F_l} \qquad \textbf{(5.9)}$$

where $F_u = F_{\alpha/2,(t-1),(N-t)}$, $F_l = F_{(1-\alpha/2),(t-1),(N-t)}$, and $F_0 = MSA/MSW$. The interval may be used for testing the hypothesis H_0: $\rho_I = 0$, where the hypothesis is not rejected if the interval includes zero (Koch, 1983).

The estimate of the intraclass correlation for castings of high-temperature alloys is

$$\widehat{\rho}_I = \frac{(73.94 - 5.82)}{\{73.94 + 9(5.82)\}} = 0.54$$

With $F_0 = 12.71$, $F_{.05,2,27} = 3.35$, and $F_{.95,2,27} = 0.051$, the 90% confidence interval estimate is

$$\frac{(12.71 - 3.35)}{\{12.71 + 9(3.35)\}} < \rho_I < \frac{(12.71 - 0.051)}{\{12.71 + 9(0.051)\}}$$

and

$$0.22 < \rho_I < 0.96$$

The interval does not include zero, and a null hypothesis of zero intraclass correlation among the bars within castings is rejected.

The interpretation of intraclass correlation can be made on the basis of the ratio in Equation (5.6). The numerator (σ_a^2) reflects the variation peculiar to the differences among groups, whereas the denominator variance ($\sigma_a^2 + \sigma_e^2$) pertains to individuals sampled randomly from the universe of all groups without regard to group boundaries.

If the intraclass correlation is large, all the individuals in the same group are affected alike by the random effect (a_i) common to that group. Thus, the similarity among individuals within groups will be greater than that among individuals from different groups, and σ_e^2 will be small relative to σ_a^2.

On the other hand, a small intraclass correlation indicates dissimilarity among individuals within groups with σ_e^2 large relative to σ_a^2. For example, competition among plants or animals within a group for nutritional resources could lead to growth disparities within a group. This could happen if more vigorous or aggressive individuals took a greater part of the nutritional resource.

5.7 Unequal Numbers of Observations in the Groups

The random effects model for the one-way classification with unequal numbers of observations per group is

$$y_{ij} = \mu + a_i + e_{ij} \qquad \textbf{(5.10)}$$

$$i = 1, 2, \ldots, t \qquad j = 1, 2, \ldots, r_i$$

with the same assumptions and interpretations given for the random model in Equation (5.1). The analysis of variance computations are the same as those for the fixed effects model with unequal replication. The expected mean squares for Among Groups and Within Groups are, respectively,

$$E(MSA) = \sigma_e^2 + r_0\sigma_a^2$$

and **(5.11)**

$$E(MSE) = \sigma_e^2$$

where

$$r_0 = \frac{1}{t-1}\left[N - \sum_{i=1}^{t}\frac{r_i^2}{N}\right] \tag{5.12}$$

The analysis of variance estimators for the variance components σ_e^2 and σ_a^2 are

$$\widehat{\sigma}_e^2 = MSW \tag{5.13}$$

and

$$\widehat{\sigma}_a^2 = \frac{(MSA - MSW)}{r_0}$$

When the r_i are unequal the confidence interval estimator for σ_a^2 in Equation (5.5) no longer applies. The interval estimator for σ_e^2 from Equation (5.4) with $(N-t)$ degrees of freedom is valid.

5.8 How Many Observations to Study Variances?

The null hypothesis of interest in the random effects model, H_0: $\sigma_a^2 = 0$, is tested with $F = MSA/MSW$, and the power of the test is

$$1 - \beta = P(F > F_{\alpha,\nu_1,\nu_2} | H_0 \text{ false}) = P(F > F_{\alpha,\nu_1,\nu_2} | \sigma_a^2 > 0)$$

When $\sigma_a^2 > 0$, the distribution of F is the central F_{ν_1,ν_2} distribution multiplied by a constant $1/\lambda^2$, where

$$\lambda^2 = 1 + \frac{r\sigma_a^2}{\sigma_e^2} \tag{5.14}$$

The power of the test can be determined from the central F distribution as

$$1 - \beta = P\left[F > \frac{1}{\lambda^2}(F_{\alpha,\nu_1,\nu_2})\right]$$

Given the number of groups (t), significance level (α), desired power ($1-\beta$), and λ, the required replication numbers can be determined from charts of power curves similar to those for the fixed effects model.

A value for λ may be determined on the basis of a desired ratio for the variance components, σ_a^2/σ_e^2, or on the basis of the standard deviation of an individual observation, σ_y. Consider Example 5.1 in which the engineer manufactured several bars of a high-temperature alloy in each of several castings. If there was no variation in the strength of the bars due to castings, then the standard deviation of a bar selected at random would be $\sigma_y = \sigma_e$. The engineer may want to detect an increase in the variability among castings (σ_a^2) that causes a certain percentage increase in σ_y. Suppose P is the fixed percentage increase in σ_y that is acceptable, and beyond which the null hypothesis would be rejected. The ratio of σ_y to σ_e expressed in terms of P when the null hypothesis is rejected is

$$\frac{\sigma_y}{\sigma_e} = \frac{\sqrt{\sigma_a^2 + \sigma_e^2}}{\sigma_e} = 1 + 0.01P$$

The necessary value for the ratio σ_a^2/σ_e^2 in Equation (5.14) is

$$\frac{\sigma_a^2}{\sigma_e^2} = (1 + 0.01P)^2 - 1 \tag{5.15}$$

Charts of power curves are given in Appendix Table X for $\alpha = .05$, .01, and selected values of ν_1 and ν_2 for the F distribution. The charts plot the power of the test, $1-\beta$, versus λ, where λ^2 is given in Equation (5.14).

Example 5.2 Castings of High-Temperature Alloys Revisited

In Example 5.1 the estimates of the variance components were $\widehat{\sigma}_e^2 = 5.82$ and $\widehat{\sigma}_a^2 = 6.81$. The estimated standard deviations are $\widehat{\sigma}_e = 2.41$ and $\widehat{\sigma}_y = 3.55$. Suppose the engineer is able to run $t = 5$ castings and wants to detect an increase in the standard deviation σ_y over σ_e of $P = 35$ with a power of at least .80 at the .05 level of significance. The required value for the ratio in Equation (5.15) is

$$\frac{\sigma_a^2}{\sigma_e^2} = [1 + 0.01(35)]^2 - 1 = 0.8225$$

so that $\lambda = \sqrt{1 + r(0.8225)}$. If a value of $r = 10$ is chosen, then $\lambda = 3$. Entering Appendix Table X for $\nu_1 = 4$ and locating the approximate position of the line for $\nu_2 = 45$ with $\alpha = .05$ the value of $1-\beta$ is between 0.8 and 0.9. Therefore, the engineer could measure ten bars in each of the five castings to detect an increase of 35% or more in the standard deviation due to the castings.

5.9 Random Subsamples to Procure Data for the Experiment

It is sometimes necessary or convenient to randomly sample subunits of the experimental units to procure the requisite data for a study. The observational unit in this case is a subsample taken from a larger experimental unit. Several plants may be sampled from a field plot for measurements of insect infestation. A serum sample is frequently split into two or more subsamples prior to spectrophotometric analysis. Several samples of paint are extracted from replicate batches of each paint formulation to test paint durability.

The subsamples introduce another random source of variability for the observations in addition to that among the experimental units. It is important to distinguish between the variation contributed by subsamples and that contributed by the experimental units. This distinction becomes important to estimation of standard

errors for treatment means and tests of hypotheses about treatments. An introduction to this distinction was given in Examples 1.1 and 1.2 in the discussion on replication in Chapter 1.

Estimates of the variance components for experimental units and for subsamples identify the amount of variation contributed by the two sources. This information is used in Section 5.10 to determine the relative number of experimental units to minimize the standard error of the treatment means or the cost of the experiment.

Example 5.3 Pesticide Residue on Cotton Plants

Applications of pesticides are often part of insect management programs used for agronomic and horticultural crops. One of the concerns following application of pesticides is the concentration of pesticide residue that remains on the plants in the field after certain periods of time. Pesticide residues are evaluated with chemical assays in the laboratory using plants sampled from field plots treated with the pesticide.

Research Hypothesis: For one particular problem the investigators hypothesized that the ability to recover pesticide residue on cotton plant leaves differed among two standard chemistry methods that were being used on a regular basis for the residue assays.

Treatment Design: The treatments consisted of the two standard chemistry methods, methods A and B, that were used on a regular basis.

Experiment Design: Six batches of plants, each batch from a single field plot, were sampled from the field and prepared for residue analysis. Three batches were randomly allocated to each of methods A and B in a completely randomized design.

The amount of plant material in each batch sampled from the field exceeded the amount required for an assay in the laboratory by either chemistry method. Thus, two subsamples of the required quantity of plant material were taken from each batch of the prepared plant tissue and analyzed by the appropriate method. Consequently, there were two chemistry methods, three batches (replications) per method, and two subsamples from each batch. The pesticide residues determined for each of the subsamples as micrograms per unit of weight are shown in Table 5.4.

The Statistical Model with Subsamples

When there are n subsamples from each of r experimental units for t treatments the statistical model is

$$y_{ijk} = \mu + \tau_i + e_{ij} + d_{ijk} \qquad \textbf{(5.16)}$$

$$i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, r \quad k = 1, 2, \ldots, n$$

where μ is the general mean, τ_i is the fixed effect of the ith treatment, e_{ij} is the random experimental error effect for the jth experimental unit of the ith treatment,

Table 5.4 Pesticide residue (μg) found on samples of cotton plants

Method A					*Method B*				
Batch	*Sample*	y_{ijk}	$\overline{y}_{ij.}$	$\overline{y}_{i..}$	*Batch*	*Sample*	y_{ijk}	$\overline{y}_{ij.}$	$\overline{y}_{i..}$
1	1	120			4	7	71		
	2	110	115.0			8	71	71.0	
2	3	120			5	9	70		
	4	100	110.0			10	76	73.0	
3	5	140			6	11	63		
	6	130	135.0	120.0		12	68	65.5	69.8
				$\overline{y}_{...} = 94.9$					

Source: G. Ware and B. Estesen, Department of Entomology, University of Arizona.

and d_{ijk} is the random effect for the kth subsample of the jth experimental unit of the ith treatment. It is assumed that the e_{ij} and d_{ijk} are normally distributed independent random effects with means 0 and variances σ_e^2 and σ_d^2, respectively. If treatments are random, then the fixed treatment effects (τ_i) in Equation (5.16) are replaced by the random group effects (a_i), which have a normal distribution with mean 0 and variance σ_a^2.

The Analysis of Variance with Subsamples

The observations, expressed as deviations from the grand mean, can be written as a sum of three separate deviations that represent the sources of variation in the experiment:

$$(y_{ijk} - \overline{y}_{...}) = (\overline{y}_{i..} - \overline{y}_{...}) + (\overline{y}_{ij.} - \overline{y}_{i..}) + (y_{ijk} - \overline{y}_{ij.}) \qquad \textbf{(5.17)}$$

The deviation of any observation from the grand mean shown on the left-hand side of Equation (5.17) is the sum of three terms. They are the

- treatment deviation $(\overline{y}_{i..} - \overline{y}_{...})$ [e.g., $(\overline{y}_{1..} - \overline{y}_{...}) = 120.0 - 94.9 = 25.1$]
- experimental error $(\overline{y}_{ij.} - \overline{y}_{i..})$ [e.g., $(\overline{y}_{11.} - \overline{y}_{1..}) = 115.0 - 120.0 = -5.0$]
- sampling error $(y_{ijk} - \overline{y}_{ij.})$ [e.g., $(y_{111} - \overline{y}_{11.}) = 120.0 - 115.0 = 5.0$]

Squaring and summing both sides of Equation (5.17) results in the total sum of squares on the left-hand side expressed as a sum of the sums of squares for treatments, experimental error, and sampling, respectively. The fundamental partition of the total sum of squares is

$$SS \text{ Total} = SS \text{ Treatment} + SS \text{ Error} + SS \text{ Sampling} \qquad \textbf{(5.18)}$$

The sums of squares partitions are summarized in the analysis of variance shown in Table 5.5 with expected mean squares for the model with fixed treatment effects.

Table 5.5 Analysis of variance for the completely randomized design with subsamples[2]

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*Expected Mean Square*
Total	$trn-1$	*SS* Total		
Treatments	$t-1$	*SST*	*MST*	$\sigma_d^2+n\sigma_e^2+rn\theta_t^2$
Error	$t(r-1)$	*SSE*	*MSE*	$\sigma_d^2+n\sigma_e^2$
Sampling	$tr(n-1)$	*SSS*	*MSS*	σ_d^2

$$SS\text{ Total}=\sum_{i=1}^{t}\sum_{j=1}^{r}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2$$

$$SST=SS\text{ Treatment}=rn\sum_{i=1}^{t}(\bar{y}_{i..}-\bar{y}_{...})^2$$

$$SSE=SS\text{ Error}=n\sum_{i=1}^{t}\sum_{j=1}^{r}(\bar{y}_{ij.}-\bar{y}_{i..})^2$$

$$SSS=SS\text{ Sampling}=\sum_{i=1}^{t}\sum_{j=1}^{r}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2$$

Two sources contribute to variation in the observations that make up the estimate of a treatment mean: the variation among the replicate experimental units treated alike (σ_e^2) and the variation among the sampling units within the same experimental units (σ_d^2). Consequently, the variance of a treatment mean is

$$\sigma_{\bar{y}_{i..}}^2=\frac{\sigma_d^2}{rn}+\frac{\sigma_e^2}{r} \tag{5.19}$$

when there are n subsamples from each of the r replicate experimental units. The standard error of any treatment mean is estimated by

$$s_{\bar{y}_{i..}}=\sqrt{\frac{MSE}{rn}} \tag{5.20}$$

The analysis of variance for the data in Table 5.4 is shown in Table 5.6. The null hypothesis of no differences among treatment effects, H_0: $\tau_i=0$, is rejected if $F_0=MST/MSE$ exceeds $F_{\alpha,(t-1),t(r-1)}$. The F_0 statistic[3] in Table 5.6 to test the null hypothesis of no difference between the means of methods A and B, $F_0=7550.08/190.08=39.72$, is exceeded with probability .003. Since the mean of method A, $\bar{y}_{1..}=120$, exceeds that of method B, $\bar{y}_{2..}=69.8$, it may be concluded that method A recovers more of the pesticide residue than method B.

[2] The sum of squares for "Error" represents the sum of squares for experimental units nested within the treatment groups. A computer program will require a term in its syntax that designates the experimental units *within* treatments.

[3] Frequently, it is necessary to specify the correct denominator for the F_0 statistic in the instructions to a computing program. By default, many programs utilize the last partition sum of squares for the denominator of the F_0 statistic for all lines of the analysis of variance table.

Table 5.6 Analysis of variance for pesticide residue from subsamples of cotton plants

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*F*	*Pr > F*
Total	11	8640.91			
Methods	1	7550.08	7550.08	39.72	.003
Error	4	760.33	190.08		
Sampling	6	330.50	55.08		

The standard error estimate for a method mean, Equation (5.20) is

$$s_{\bar{y}_{i..}}=\sqrt{\frac{190.08}{6}}=5.63$$

and the standard error estimate of the difference between the two method means is

$$s_{(\bar{y}_{i..}-\bar{y}_{j..})}=\sqrt{\frac{2(190.08)}{6}}=7.96$$

5.10 Using Variance Estimates to Allocate Sampling Efforts

The distribution of resources at the planning stage of an experiment involving subsamples requires decisions regarding the number of experimental units to use and the number of subsamples to take from each experimental unit. The objective is to have a design that results in greater precision—a smaller variance for the estimate of a treatment mean ($\sigma_{\bar{y}_{i..}}^2$) for a fixed cost. When estimates of the variance components and relative costs of the experimental and sampling units are available it is possible to provide an optimum allocation of effort between experimental units and sampling units in the experiment.

Cochran (1965) provided an optimum allocation solution based on the cost function $C=c_1r+c_2rn$. The value of C is the cost for a single treatment in the experiment composed of r experimental units each at a cost of c_1 and rn sampling units each at a cost of c_2. The objective may be posed as an attainment of minimum cost (C) for a fixed variance in Equation (5.19), or the attainment of a minimum variance for a fixed cost. Either way the solution for the number of sampling units (n) is

$$n=\sqrt{\frac{c_1\sigma_d^2}{c_2\sigma_e^2}} \tag{5.21}$$

The value of r is found by solving the cost equation for r if the cost is fixed or by solving the variance Equation (5.19) if the variance is fixed.

Example 5.4 Pesticide Residue Revisited

An optimum allocation for plots of cotton plants and subsamples per plot is required for pesticide residue studies. The estimates of the variance components obtained from Table 5.6 are

$$\widehat{\sigma}_d^2 = MSS = 55.08$$

and

$$\widehat{\sigma}_e^2 = \frac{(MSE - MSS)}{n} = \frac{(190.08 - 55.08)}{2} = 67.50$$

Suppose the cost of one plot is $c_1 = 1.0$ relative to the cost of preparing and analyzing one subsample, $c_2 = 0.1$. The estimated number of subsamples per plot is

$$n = \sqrt{\frac{1(55.08)}{0.1(67.50)}} = 2.86$$

Three subsamples would be required from each plot. If the investigator desired a standard error for the treatment mean of $\sigma_{\bar{y}_{i\cdot}} = 3$ or a variance of 9, the number of required plots r can be found from the substitution of the required quantities in Equation (5.19). The substitutions are

$$\widehat{\sigma}^2_{\bar{y}_{i\cdot}} = 9,\ \widehat{\sigma}_e^2 = 67.5,\ \widehat{\sigma}_d^2 = 55.08, \text{ and } n = 3$$

so that

$$9 = \frac{55.08}{r \cdot 3} + \frac{67.5}{r};\ \ 9r = 85.86;\ \ r = 9.54$$

The investigator would have to use ten plots with three subsamples per plot to have a standard error of treatment means equal to 3 with relative costs of 1.0 and 0.1 for plots and subsamples, respectively.

5.11 Unequal Numbers of Replications and Subsamples

Unequal subsample and replication numbers can occur in a study. The three different possibilities are (1) unequal numbers of experimental units per treatment with unequal numbers of subsamples per experimental unit, (2) equal numbers of experimental units per treatment with unequal numbers of subsamples per experimental unit, and (3) unequal numbers of experimental units per treatment with equal numbers of subsamples per experimental unit. Any imbalance in the number of observations at the experimental unit or subsample stage affects the computations for the analysis of variance and the expected mean squares.

Example 5.5 Biology of the Tobacco Budworm

Populations of insects often develop resistance to the toxic effects of an insecticide after long-term exposure to the insecticide. When this resistance develops the insecticide is no longer effective to control the population below levels harmful to the crop.

Populations of the tobacco budworm, an insect pest harmful to the cotton plant, have developed resistance to a number of common insecticides. The insecticides are one component of the overall program of insect control in crops. Other components of the control program are also dependent on the biology of the insects in terms of their reproductive life cycle and developmental patterns.

Research Hypothesis: Entomologists hypothesized that the development of insecticide resistance could also affect other aspects of the tobacco budworm's biology. If this were true, then the changes in the insect's biology would have an effect on tobacco budworm control programs.

Treatment Design: The treatments used to address the research hypothesis included three strains of the tobacco budworm: (1) USDA, a strain very susceptible to a pyrethroid insecticide; (2) Resistant, a strain quite resistant to the insecticide; and (3) Field, a naturally occurring strain collected in a local cotton field. Both the Resistant and USDA strains were populations maintained in artificial environments to sustain their resistance characteristics. Any differences in the biology of the two strains were considered reflective of changes associated with developed insecticide resistance. The biology of the naturally occurring Field strain served as a control treatment for this supposition. One of the characteristics measured to evaluate the biology was the weight of male larvae.

Experiment Design: Six random matings between female and male moths were made from each of the strains, and the offspring from each mating were reared in separate enclosures in the laboratory. The 18 enclosures were placed randomly within the rearing facility. Unequal numbers of offspring resulted among the 18 matings as shown with the data listed in Table 5.7.

The Statistical Model and Analysis

Suppose the experiment has r_i experimental units for the ith treatment group and n_{ij} subsamples for the jth experimental unit of the ith treatment. The statistical model for unequal subsamples is

$$y_{ijk} = \mu + \tau_i + e_{ij} + d_{ijk} \qquad \textbf{(5.22)}$$

$$i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, r_i \quad k = 1, 2, \ldots, n_{ij}$$

Table 5.7 Weight of male larvae from six matings in each of three tobacco budworm strains

Strain	*Mating*	*Weight*	n_{ij}	$n_{i.}$	$\overline{y}_{ij.}$	$\overline{y}_{i..}$
USDA	1	305, 300	2		302.5	
	2	376, 363, 389	3		376.0	
	3	282	1		282.0	
	4	309, 321	2		315.0	
	5	354, 308, 327	3		329.7	
	6	330	1	12	330.0	330.3
Field	7	280	1		280.0	
	8	311, 349, 291, 286	4		309.3	
	9	377, 342	2		359.5	
	10	346, 340, 347	3		344.3	
	11	360	1		360.0	
	12	359, 299	2	13	329.0	329.8
Resistant	13	273, 276	2		274.5	
	14	272, 253	2		262.5	
	15	315, 262, 297	3		291.3	
	16	323	1		323.0	
	17	252	1		252.0	
	18	319; 298	2	11	308.5	285.5
					$\overline{y}_{...} = 316.4$	

Source: Dr. T. Watson and S. Kelly, Department of Entomology, University of Arizona.

where μ is the general mean, τ_i is the fixed effect of the ith treatment, e_{ij} is the random experimental error effect for the jth experimental unit of the ith treatment, and d_{ijk} is the random effect for the kth subsample of the jth experimental unit of the ith treatment. We assume the e_{ij} and d_{ijk} are normally distributed independent random effects with means 0 and variances σ_e^2 and σ_d^2, respectively. If treatments are random, then the fixed treatment effects (τ_i) in Equation (5.22) are replaced by the random group effects (a_i), which have a normal distribution with mean 0 and variance σ_a^2.

The sums of squares partitions for the analysis of variance and expected mean squares for random treatment effects are shown in Table 5.8. If treatment effects are fixed, replace $c_2\sigma_a^2$ with $\sum n_{ij}(\mu_i - \overline{\mu}_.)^2/(t-1)$.

Notice that the squares of the treatment mean deviations in SST are weighted by the number of observations on the treatments, $n_{i.}$, and the squares of the experimental error deviations in SSE are weighted by the number of subsamples for the experimental units, n_{ij}. The coefficients for the variance components in the expected mean squares with random treatment effects are

Table 5.8 Analysis of variance for the completely randomized design with unequal numbers of replications and subsamples

Source of Variation	*Degree of Freedom*	*Sum of Squares*	*Mean Square*	*Expected Mean Square*
Total	$N-1$	*SS* Total		
Treatments	$t-1$	SST	MST	$\sigma_d^2 + c_1\sigma_e^2 + c_2\sigma_a^2$
Error	$\sum_{i=1}^{t} r_i - t$	SSE	MSE	$\sigma_d^2 + c_3\sigma_e^2$
Sampling	$N - \sum_{i=1}^{t} r_i$	SSS	MSS	σ_d^2

$$N = \sum_{i=1}^{t}\sum_{j=1}^{r_i} n_{ij} \qquad n_{i.} = \sum_{j=1}^{r_i} n_{ij}$$

$$SS\ \text{Total} = \sum_{i=1}^{t}\sum_{j=1}^{r_i}\sum_{k=1}^{n_{ij}} (y_{ijk} - \overline{y}_{...})^2$$

$$SST = \sum_{i=1}^{t} n_{i.}(\overline{y}_{i..} - \overline{y}_{...})^2$$

$$SSE = \sum_{i=1}^{t}\sum_{j=1}^{r_i} n_{ij}(\overline{y}_{ij.} - \overline{y}_{i..})^2$$

$$SSS = \sum_{i=1}^{t}\sum_{j=1}^{r_i}\sum_{k=1}^{n_{ij}} (y_{ijk} - \overline{y}_{ij.})^2$$

$$c_1 = \frac{1}{t-1}\left(A - \frac{B}{N}\right), c_2 = \frac{1}{t-1}\left(N - \frac{D}{N}\right), \text{ and } c_3 = \frac{1}{\left(\sum_{i=1}^{t} r_i - t\right)}(N - A) \tag{5.23}$$

where

$$A = \sum_{i=1}^{t}\sum_{j=1}^{r_i}\left(\frac{n_{ij}^2}{n_{i.}}\right), \quad B = \sum_{i=1}^{t}\sum_{j=1}^{r_i} n_{ij}^2, \text{ and } D = \sum_{i=1}^{t} n_{i.}^2$$

When the number of subsamples are equal for each of the experimental units, then $n_{ij} = n$ for all i and j; the coefficients are

$$c_1 = c_3 = n \qquad \text{and} \qquad c_2 = \frac{1}{t-1}\left[N - \frac{D}{N}\right] \tag{5.24}$$

When $c_1 = c_3 = n$ the expected mean squares for Treatments and Error will be identical under the null hypothesis H_0: $\tau_i = 0$, and the statistic $F_0 = MST/MSE$ is used to test the hypothesis.

The analysis of variance for the data is shown in Table 5.9. The expected mean squares for analysis of variance are also shown in Table 5.9 for random treatment effects. The calculations of the coefficients for the variance components are shown in Appendix 5A.

Table 5.9 Analysis of variance for male larval weights from six matings in each of three tobacco budworm strains

Source of Variation	*Degrees of Freedom*	*Sum of Squares*	*Mean Square*	*Expected Mean Square*
Total	35	46516.75		
Strain	2	15187.05	7593.52	$\sigma_d^2 + 2.36\sigma_e^2 + 11.97\sigma_a^2$
Error	15	23082.45	1538.83	$\sigma_d^2 + 1.93\sigma_e^2$
Sampling	18	8247.25	458.18	σ_d^2

Tests of Hypotheses Require Approximate *F* Tests

When the subsample numbers are not equal, c_1 and c_3 can have different values. There is no exact test of the null hypothesis for treatment effects because no two mean squares have the same expected mean squares under the null hypothesis if c_1 and c_3 have different values. An approximate F_0 statistic can be calculated to test the null hypothesis of no treatment effects when $c_1 \neq c_3$. An approximate test is necessary for the tobacco budworm experiment since $c_1 = 2.36$ and $c_3 = 1.93$ in Table 5.9.

A Mean Square for Error is devised with an expectation equal to that of the Mean Square for Treatments, given a true null hypothesis with $E(MST) = \sigma_d^2 + c_1\sigma_e^2$.

The required mean square is constructed with a linear function of MSS and MSE as

$$M = a_1 MSE + a_2 MSS \tag{5.25}$$

If $a_1 = c_1/c_3$ and $a_2 = 1 - c_1/c_3$, the expected value of M will be $\sigma_d^2 + c_1\sigma_e^2$ as required for the approximate F_0 test.

The Satterthwaite Approximation for Degrees of Freedom

Satterthwaite (1946) derived the following result for a linear function of mean squares. Given a linear function M, where

$$M = a_1 MS_1 + a_2 MS_2 + \cdots + a_k MS_k \tag{5.26}$$

and $MS_1, MS_2, \ldots, MS_k$ are mean squares with degrees of freedom $\nu_1, \nu_2, \ldots \nu_k$, respectively, the degrees of freedom for M are approximated by

$$\nu = \frac{M^2}{\sum_{i=1}^{k} \frac{(a_i MS_i)^2}{\nu_i}} \tag{5.27}$$

The linear function of mean squares necessary to test the hypothesis of no difference in mean larval weights among the three tobacco budworm strains requires

$$a_1 = \frac{c_1}{c_3} = \frac{2.36}{1.93} = 1.22 \quad \text{and} \quad a_2 = 1 - \frac{c_1}{c_3} = 1 - 1.22 = -0.22$$

$$MSE = 1538.83 \text{ with } 15 \text{ d.f.} \quad \text{and} \quad MSS = 458.18 \text{ with } 18 \text{ d.f.}$$

From Equation (5.25)

$$M = a_1 MSE + a_2 MSS = 1.22(1538.83) - 0.22(458.18) = 1776.57$$

The degrees of freedom for M from Equation (5.27) are

$$\nu = \frac{1776.57^2}{\frac{[1.22(1538.83)]^2}{15} + \frac{[-0.22(458.18)]^2}{18}} = 13.4$$

The truncated value of $\nu = 13$ is used as the degrees of freedom for M. The value of the test statistic is $F_0 = MST/M = 7593.53/1776.57 = 4.27$. The critical value at the $\alpha = .05$ level of significance is $F_{.05,2,13} = 3.81$, and the null hypothesis is rejected. There are some differences among the mean larval weights of the three strains. The test is only approximate, and it should be noted that the approximation is degraded somewhat if some of the coefficients (a_i) in Equation (5.26) are negative.

EXERCISES FOR CHAPTER 5

1. A genetics study with beef animals consisted of several sires each mated to a separate group of dams. The matings that resulted in male progeny calves were used for an inheritance study of birth weights. The birth weights of eight male calves in each of five sire groups follow.

Sire	*Birthweights*
177	61, 100, 56, 113, 99, 103, 75, 62
200	75, 102, 95, 103, 98, 115, 98, 94
201	58, 60, 60, 57, 57, 59, 54, 100
202	57, 56, 67, 59, 58, 121, 101, 101
203	59, 46, 120, 115, 115, 93, 105, 75

Source: Dr. S. DeNise, Department of Animal Sciences, University of Arizona.

a. Assume a random model for this study. Write the linear model, explain each of the terms, compute the complete analysis of variance, and show the expected mean squares.
b. Estimate the components of variance for sires and progeny within sires, and determine the 90% confidence interval estimates.
c. Test the null hypothesis H_0: $\sigma_a^2 = 0$ for the sires.
d. Estimate the intraclass correlation coefficient, and give the 90% confidence interval estimate.

2. The data from Exercise 3.5 are cholesterol concentrations from laboratory analyses of 2 samples from each of 8 patients.
a. Assume a random model for the study. Write a linear model, explain each of the terms, compute the analysis of variance, and show the expected mean squares.
b. Estimate the components of variance for patients and samples and determine the 90% confidence interval estimates.
c. Estimate the intraclass correlation coefficient and give the 90% confidence interval estimate.
d. What is the interpretation of the intraclass correlation coefficient in this study?

3. Think of research problems in your field of interest for which the treatments in the study could be a random sample from a large population of treatments.
a. Describe a particular study you could conduct.
b. Describe how you would conduct the study.
c. Write the linear model for your study; identify the terms; and write out the analysis of variance table showing sources of variation, degrees of freedom, and expected mean squares.
d. Explain why it would be important to know the magnitude of the Among Group and Within Group components of variance.
e. Describe how you would use estimates of the components of variance.
f. What assumptions do you have to make about your study to have valid inferences from your variance component estimates?

4. A plant pathologist took four 3-pound samples from 50-ton lots of cottonseed accumulated at various cotton gins during the ginning season. The samples of seed were analyzed in the laboratory for Aflatoxin, which is a toxin produced by organisms associated with the seeds. The Aflatoxin concentrations in parts per billion for samples from eight lots of cottonseed follow.

Lot Number	*Aflatoxin (ppb)*
3469 – 72	39, 57, 63, 66
3849 – 52	56, 13, 25, 31
3721 – 24	64, 83, 88, 71
3477 – 80	29, 55, 21, 51
3669 – 72	38, 66, 53, 81
3873 – 76	11, 49, 34, 10
3777 – 80	23, 0, 5, 20
3461 – 64	10, 11, 23, 37

Source: Dr. T. Russell, Department of Plant Pathology, University of Arizona.

a. Assume lots and samples within lots are random effects. Write the linear model for the study, explain the terms, compute the complete analysis of variance, and show the expected mean squares.
b. Estimate the components of variance for lots and samples within lots.
c. What is the total variance estimate $(\widehat{\sigma}_y^2)$ for an individual observation?
d. What proportion of the total variation (σ_y^2) in Aflatoxin can be attributed to variation among lots and samples within lots, respectively?
e. What is the standard deviation estimate $(\widehat{\sigma}_y)$ for an individual observation?
f. Explain how the variance component estimates might be used to plan future sampling for Aflatoxin contamination.

5. Think of research problems in your field of interest that require you to take samples of the experimental (or observational) unit because the unit cannot be measured in its entirety.
a. Describe a specific study you could conduct.
b. Describe how you would conduct the study.
c. Write a linear model for your study; identify the terms; and sketch the analysis of variance showing sources of variation, degrees of freedom, and expected mean squares.
d. What would be the relative costs for experimental units (c_1) and sampling units (c_2)?

6. A study was conducted on high-energy particulate cartridge filters used with commercial respirators for protection against particulate matter. One particular test included three filters randomly selected from each of two manufacturers. Three independent replicate tests were made on each of the filters. The measurements were the percent penetration by a standard type of test aerosol.

	Manufacturer 1			*Manufacturer 2*		
Filter	*1*	*2*	*3*	*4*	*5*	*6*
	1.12	0.16	0.15	0.91	0.66	2.17
	1.10	0.11	0.12	0.83	0.83	1.52
	1.12	0.26	0.12	0.95	0.61	1.58

Source: R. J. Beckman and C. J. Nachtsheim (1987), Diagnostics for mixed-model analysis of variance, *Technometrics* 29, 413–426.

a. Write a linear model for this study, explain each of the terms, compute the analysis of variance, and show the expected mean squares.
b. Test the hypothesis that there is no difference between the average percent penetration of the filters for the two manufacturers.
c. Compute the means, their standard errors, and the 95% confidence interval estimates of the means for each of the manufacturers.
d. Suppose the relative costs, c_1:c_2, for the study are 200:1, where c_1 is the cost of a filter and c_2 is the cost of an independent filter test. The engineers wanted to achieve a standard error for a mean of 0.20. How many filters and how many tests per filter would be required?

7. A soil scientist studied the growth of barley plants under three different levels of salinity in a controlled growth medium. There were two replicate containers for each treatment in a completely randomized design and three plants were measured in each replication. The data on the dry weight of the plants in grams follow.

Salinity	*Container*	*Weight(g)*
Control	1	11.29, 11.08, 11.10
	2	7.37, 6.55, 8.50
6 Bars	3	5.64, 5.98, 5.69
	4	4.20, 3.34, 4.21
12 Bars	5	4.83, 4.77, 5.66
	6	3.28, 2.61, 2.69

Source: Dr. T. C. Tucker, Department of Soil and Water Science, University of Arizona.

a. Write a linear model for an analysis of the data, explain the terms, compute the analysis of variance, and show the expected mean squares.
b. Test the hypothesis of no difference among the means of the salinity levels.
c. Compute the standard error of a salinity level mean.
d. Partition the sum of squares for salinity into two orthogonal polynomial sums of squares (linear and quadratic), each with 1 degree of freedom, and test the null hypotheses of no linear or quadratic regression.
e. Suppose the relative costs, c_1:c_2 are 10:0.1, where c_1 is the cost of setting up and maintaining another replicate container and c_2 is the cost of measuring the weights in a container. How many replicate containers and plants per container would be necessary to achieve a standard error for a treatment mean of 0.75?

8. The porosity index is a measure used by soil scientists to assist in the prediction of water movement, storage, availability, and aeration conditions of soils. A soil scientist utilized a special sampling design to take soil samples from one of the university experiment farms to measure the porosity index of the farm soil. The farm was partitioned into fields of approximately 4 hectares each divided into eight sections. The sampling plan included a random selection of fields from which sections were randomly selected. Locations for soil subsamples were randomly selected within the sections. A special staggered sampling design from Goldsmith and Gaylor (1970) was utilized for the study. The porosity index of each soil subsample follows.

Field	*Section*	*Porosity*	*Field*	*Section*	*Porosity*
1	1	3.846, 3.712	9	17	5.942
	2	5.629, 2.021		18	5.014
2	3	5.087	10	19	5.143
	4	4.621		20	4.061
3	5	4.411	11	21	3.835, 2.964
	6	3.357		22	4.584, 4.398
4	7	3.991	12	23	4.193
	8	5.766		24	4.125
5	9	5.677	13	25	3.074
	10	3.333		26	3.483
6	11	4.355, 6.292	14	27	3.867
	12	4.940, 4.810		28	4.212
7	13	2.983	15	29	6.247
	14	4.396		30	4.730
8	15	5.603			
	16	3.683			

Source: Dr. A. Warrick and M. Coelho, Department of Soil and Water Science, University of Arizona.

a. Assume all effects are random. Write a linear model for the study, explain each of the terms, compute the analysis of variance for the data, and show the expected mean squares.
b. Estimate the components of variance for fields, sections, and samples.
c. Test the null hypothesis H_0: $\sigma_a^2 = 0$ for the fields' component of variance.
d. Test the null hypothesis H_0: $\sigma_e^2 = 0$ for the sections' component of variance.

9. Use the data from Exercise 5.3 to determine how many samples the plant pathologist would have to take from each of five lots of cottonseed to detect a ratio $\sigma_a^2/\sigma_e^2 = 2$ at the .01 significance level with a power of .90.

5A Appendix: Coefficient Calculations for Expected Mean Squares in Table 5.9

$$A = \sum_{i=1}^{t}\sum_{j=1}^{r_i}\left(\frac{n_{ij}^2}{n_{i.}}\right)$$

$$= \frac{2^2 + \cdots + 1^2}{12} + \frac{1^2 + \cdots + 2^2}{13} + \frac{2^2 + \cdots + 2^2}{11} = 7.11655$$

$$B = \sum_{i=1}^{t}\sum_{j=1}^{r_i} n_{ij}^2 = 2^2 + 3^3 + \cdots + 1^2 + 2^2 = 86$$

$$D = \sum_{i=1}^{t} n_{i.}^2 = 12^2 + 13^2 + 11^2 = 434$$

$$c_1 = \frac{1}{t-1}\left(A - \frac{B}{N}\right) = \frac{1}{2}\left(7.11655 - \frac{86}{36}\right) = 2.36$$

$$c_2 = \frac{1}{t-1}\left(N - \frac{D}{N}\right) = \frac{1}{2}\left(36 - \frac{434}{36}\right) = 11.97$$

$$c_3 = \frac{1}{\left(\sum_{i=1}^{t} r_i - t\right)}(N - A) = \frac{1}{15}(36 - 7.11655) = 1.93$$

6 Factorial Treatment Designs

The factorial treatment design was introduced in Chapter 1 as a way to investigate the relationships among several types of treatments. The basic factorial treatment design in a completely randomized experiment design and its analysis are introduced in this chapter. Planned contrasts and response curve estimation, discussed in Chapter 3, are applied to the factorial treatment design. Methods to determine the number of required replications and to analyze the factorial treatment design with one replication or unequal treatment replications are discussed as well.

6.1 Efficient Experiments with Factorial Treatment Designs

Comparisons among treatments can be affected substantially by the conditions under which they occur. Frequently, clear interpretations of effects for one treatment factor must take into account the effects of other treatment factors. A special type of treatment design, **factorial treatment design**, was developed to investigate more than one factor at a time.

Factorial treatment designs produce efficient experiments. Each observation supplies information about all of the factors, and we are able to look at responses to one factor at different levels of another factor in the same experiment. The response to any factor observed under different conditions indicates whether the factors act on the experimental units independently of one another. Interaction between factors occurs when they do not act independently of one another.

Example 6.1 Compaction Effects on Asphaltic Concrete Durability

Asphalt pavements undergo water-associated deteriorations such as cracking, potholes, and surface raveling. The weakened pavement occurs when there is