Since condition $\Sigma_i \widehat{\tau}_i = 0$ has been imposed on the solution, the substitution

$$-\widehat{\tau}_i = \sum_{\substack{p=1 \\ p \neq i}}^{t} \widehat{\tau}_p$$

can be made in Equation (9A.8). With the equality $\lambda(t-1) = r(k-1)$, the equation for $\widehat{\tau}_i$ is

$$\lambda t \widehat{\tau}_i = kQ_i \tag{9A.9}$$

$$\widehat{\tau}_i = \frac{kQ_i}{\lambda t} \quad i = 1, 2, \ldots, t$$

10 Incomplete Block Designs: Resolvable and Cyclic Designs

A general description of and analyses for incomplete block designs were introduced in Chapter 9. Several major classes of useful incomplete block designs illustrated in this chapter include resolvable designs with blocks grouped in complete replications of treatments. Cyclic designs that can be constructed without the use of extensive tabled plans and the α designs that extend the number of available resolvable designs for experiments are also discussed.

10.1 Resolvable Designs to Help Manage the Experiment

Resolvable designs have blocks grouped such that each group of blocks constitutes one complete replication of the treatments. The grouping into complete replications is useful in the management of an experiment.

One of the early applications for resolvable designs occurred with plant-breeding trials placed in field plots on experimental farmland. Plant breeders wanted to test a large number of genetic lines and to make all comparisons among pairs of lines with equal precision. Arrangement of field plots into blocks smaller than a complete replication was necessary to reduce experimental error variance more so than was possible with the complete block designs. The resolvable incomplete block design was attractive because it not only reduced the block sizes for greater precision, but it also allowed the researcher to manage these large studies in the field on a replication-by-replication basis.

The resolvable designs can be useful in practice when the entire experiment cannot be completed at one time. The experiment in a resolvable design can be conducted in stages, with one or more replications completed at each stage. In addition, should the experiment for any reason be terminated prematurely there will be equal replication of all treatments.

The resolvable designs are arranged in r replicate groups of s blocks with k units per block. The number of treatments is a multiple of the number of units per block, $t = sk$, and the total number of blocks satisfies the relationship $b = rs \geq t + r - 1$ for balanced resolvable incomplete block designs.

Example 10.1 A Resolvable Design for Food Product Tests

Consider an experiment to test nine variations of a food product on the same day with human subjects as judges. Several factors associated with this experiment make an incomplete block design attractive. First, a judge can be expected to adequately discriminate at most four or five food product samples at any one time. Due to scheduling problems, it is not possible to have a sufficient number of judges available on any one day for adequate replication of the treatments. New food preparations must be made each day of the test, and it is necessary to ensure day-to-day variation in food preparation does not interfere with treatment comparisons in the experiment.

A test designed for three judges to each evaluate three of the food product variations on a given day is shown in Display 10.1. Three judges are scheduled on each of four test days. The three judges each constitute one incomplete block of three treatments. Each judge is randomly assigned to one of the blocks of three treatments and randomly presented with three food products for evaluation.

The design is resolvable with one complete replication of the experiment conducted on any given day. The design is balanced since each treatment occurs with every other treatment one time in the same block somewhere in the experiment. The judges only evaluate three food products at one time, reducing the within-block variation considerably from what would have resulted if a judge been required to evaluate nine products at one time.

Display 10.1 A Resolvable Balanced Incomplete Block Design to Evaluate Food Products

Judge	*Day I*	Judge	*Day II*	Judge	*Day III*	Judge	*Day IV*
1	(1, 2, 3)	4	(1, 4, 7)	7	(1, 5, 9)	10	(1, 6, 8)
2	(4, 5, 6)	5	(2, 5, 8)	8	(2, 6, 7)	11	(2, 4, 9)
3	(7, 8, 9)	6	(3, 6, 9)	9	(3, 4, 8)	12	(3, 5, 7)

Balanced Lattice Designs

The **balanced lattice designs** are a well-known group of resolvable designs introduced by Yates (1936b). The number of these square lattice designs available is limited because the number of treatments must be an exact square $t = k^2$. The designs require $r = (k + 1)$ replications and $b = k(k + 1)$ blocks for complete balance with $\lambda = 1$. Each replicate group contains $s = k$ blocks of k experimental units each. The number of units per block k must be a prime number or a power of a prime number. Plans for balanced lattice designs with $t = 9, 16, 25, 49, 64,$ and 81 can be found in Cochran and Cox (1957) or Petersen (1985).

The design in Display 10.1 is a balanced lattice with nine treatments in blocks of three units. The design has four replication groups for a total of 12 blocks. Note that each treatment pair occurs together in the same block $\lambda = 1$ time somewhere in the balanced design.

Complete balance requires $(k + 1)$ replications with the lattice designs. Since they are resolvable designs, one or more of the replicate groups can be eliminated to produce a partially balanced lattice design. Designs with $r = 2$, 3, or 4 replicate groups are known as *simple, triple,* or *quadruple* lattices, respectively.

The average efficiency factor for a balanced lattice design is

$$E = \frac{(k+1)(r-1)}{r(k+2)-(k+1)}$$

and for simple and triple lattices the average efficiency factors are, respectively, $(k + 1)/(k + 3)$ and $(2k + 2)/(2k + 5)$.

Rectangular Lattice Designs

The restriction on numbers of treatments or varieties with square lattices is quite severe. The **rectangular lattice designs** developed by Harshbarger (1949) provided resolvable designs with treatment numbers intermediate to those provided by the square lattice designs.

The rectangular lattice designs accommodate $t = s(s - 1)$ treatments in blocks of size $k = (s - 1)$ units. The treatment numbers fall about midway between those provided by the square lattices. Cochran and Cox (1957) have tabled plans for rectangular lattices with $t = 12$, 20, 30, 42, 56, 72, and 90 treatments. Designs with $r = 2$ and $r = 3$ replicates are known as *simple* and *triple* rectangular lattices, respectively.

Other resolvable balanced incomplete block designs exist for numbers of treatments without the restrictions of the square or rectangular lattices; however, they exist for a limited set of block sizes and treatment numbers. Some balanced resolvable designs for $t \leq 11$ treatments are cataloged in Appendix 9A.1. Other balanced resolvable designs for $t > 11$ are given by Cochran and Cox (1957).

Analysis Outline for Resolvable Designs

The linear model for the resolvable design is

$$y_{ijm} = \mu + \tau_i + \gamma_j + \rho_{m(j)} + e_{ijm} \qquad (10.1)$$

$$i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, r \quad m = 1, 2, \ldots, s$$

where μ is the general mean, τ_i is the treatment effect, γ_j is the replicate group effect, $\rho_{m(j)}$ is the block nested within replication effect, and e_{ijm} is the random experimental error.

The sequential fit of alternative models outlined in Chapter 9 provides an orthogonal sum of squares partition. An additional sum of squares partition is required for replicate groups. The blocks are nested within replicate groups and the sums of squares for blocks from the replicate groups are pooled as *SS*(Blocks/Reps). The intrablock analysis of variance outline is shown in Table 10.1.

Table 10.1 Intrablock analysis for a resolvable balanced incomplete block design

Source of Variation	*Degrees of Freedom*	*Sum of Squares*
Total	$N-1$	$\sum(y_{ijm} - \bar{y}_{...})^2$
Replications	$r-1$	$sk\sum(\bar{y}_{.j.} - \bar{y}_{...})^2$
Blocks (unadjusted) within replications	$r(s-1)$	$k\sum(\bar{y}_{.jm} - \bar{y}_{...})^2$
Treatments (adjusted)	$t-1$	$\dfrac{k\sum Q_i^2}{\lambda t}$
Error	$N-t-rs+1$	By subtraction

$Q_i = y_{i..} - (B_i/k)$, and B_i is the sum of block totals that include the ith treatment.

The treatments are orthogonal to the replicate groups since each treatment appears one time in each of the replicate groups. The treatment totals adjusted only for block effects provide unbiased estimates of the treatment means and a valid F test for treatment effects.

10.2 Resolvable Row–Column Designs for Two Blocking Criteria

Some research settings require two blocking criteria and do not permit complete blocks of treatments for either the row or column blocks required by the row orthogonal Youden squares. Row–column designs have the t treatments placed into blocks of $k = pq$ units. The treatments in each block are then arranged in a $p \times q$ row–column design.

A limited number of balanced row–column incomplete block designs exists for experiments that require block sizes of both rows and columns to be less than the number of treatments. The designs require a large number of blocks to meet the requirement of balance. See John (1971) for an extended bibliography of designs.

Example 10.2 A Nested Row–Column Design to Sample Insect Populations

Consider an epidemiological study on the transmission of a microbial parasite to humans by an insect in an agricultural area of the tropics. The microbial parasite is passed to the human when the insect bites the human. In turn, if the insect is not infected by the parasite it may become infected when it bites an infected human. A public health project is planning to test the effectiveness of a drug that is known to disrupt the parasite's life cycle in humans. An entomologist working with the project is going to sample the insect populations to monitor the effect the program has on reducing their infection rate.

Nine types of habitats in and around each of four agricultural plantations are scheduled for monitoring in the project. The habitats consist of locations such as flowing streams, villages on the plantation, the agricultural fields, stagnant water ponds, and so forth. The entomologist can only collect insect samples at three such habitat sites in a single day. A row–column sampling design is set up to control the potential variation caused by sampling on different days and different times of day at each of four plantations. The design is shown in Display 10.2. The design has nine habitat treatments sampled in a 3×3 row–column design at four plantations. The three rows in the 3×3 array are the time of day the samples are collected, and the three columns are the three days for collection at each plantation.

Upon inspection of each habitat treatment pair it can be verified that each is sampled once at the same time and once on the same day in the design. Each treatment pair must occur together one time in the same row *and* one time in the same column somewhere in the design to have a balanced row–column design. The design is a resolvable design with plantations as replications and one complete replication of the nine habitat types at each plantation. It is a nested design wherein the rows and columns are nested within the replicate groups.

Balanced Lattice Square Designs

These designs, introduced by Yates (1940b), are resolvable nested row–column designs developed early in the history of incomplete block designs. The example exhibited in Display 10.2 is such a design. The **balanced lattice square** has $t = k^2$ treatments laid out in a $k \times k$ row–column design. One $k \times k$ array consists of one complete replication of the treatments with $p = q = k$. The balanced design requires $r = (k+1)$ replications for complete balance such that every treatment pair occurs together one time in the same row *and* one time in the same column

Display 10.2 A Nested 3 × 3 Row–Column Incomplete Block Design to Sample Insect Populations

Time	*Day* (1)	(2)	(3)
(1)	1	2	3
(2)	4	5	6
(3)	7	8	9

Plantation I

Time	*Day* (4)	(5)	(6)
(4)	1	4	7
(5)	2	5	8
(6)	3	6	9

Plantation II

Time	*Day* (7)	(8)	(9)
(7)	1	6	8
(8)	9	2	4
(9)	5	7	3

Plantation III

Time	*Day* (10)	(11)	(12)
(10)	1	9	5
(11)	6	2	7
(12)	8	4	3

Plantation IV

somewhere in the experiment. When k is an odd value, a replication value of $r = \frac{1}{2}(k+1)$ will provide semi-balance such that any pair of treatments occurs together one time in the same row *or* one time in the same column.

Plans for lattice square designs with $t = 9$, 16, 25, 49, 64, 81, 121, and 169 may be found in Cochran and Cox (1957) and Petersen (1985). Extensive discussions of the designs are found also in Kempthorne (1952) and Federer (1955).

Analysis Outline for Nested Row–Column Designs

The linear model for any resolvable nested row–column design is

$$y_{ijlm} = \mu + \beta_m + \rho_{j(m)} + \gamma_{l(m)} + \tau_i + e_{ijlm} \tag{10.2}$$

$$i = 1, 2, \ldots, t \quad j = 1, 2, \ldots, p \quad l = 1, 2, \ldots, q \quad m = 1, 2, \ldots, r$$

where μ is the general mean, τ_i is the treatment effect, β_m is the replicate group effect, $\rho_{j(m)}$ and $\gamma_{l(m)}$ are, respectively, the row and column nested in replicate group effect, and e_{ijlm} is the random experimental error.

Each of the treatments occurs in each replicate group, and the treatments are orthogonal to replicate groups. The rows and columns also are orthogonal to replicate groups. The sums of squares partitions for replications, rows within replications, and columns within replications can be computed in the usual manner, as shown in the analysis of variance outline in Table 10.2. The treatments are not orthogonal to rows or columns of the replicate groups. The treatments must be adjusted for both row and column effects.

The adjusted treatment totals required for least squares estimates of treatment effects and the adjusted treatment sum of squares are

Table 10.2 Intrablock analysis for a nested row–column incomplete block design

Source of Variation	*Degrees of Freedom*	*Sum of Squares*
Total	$N-1$	$\sum(y_{ijlm} - \overline{y}_{....})^2$
Groups (replications)	$r-1$	$t\sum(\overline{y}_{...m} - \overline{y}_{....})^2$
Columns (unadjusted) within groups	$r(q-1)$	$p\sum(\overline{y}_{..lm} - \overline{y}_{...m})^2$
Rows (unadjusted) within groups	$r(p-1)$	$q\sum(\overline{y}_{.j.m} - \overline{y}_{...m})^2$
Treatments (adjusted)	$t-1$	$\dfrac{pq(t-1)}{rt(p-1)(q-1)}\sum Q_i^2$
Error	$r(p-1)(q-1)-(t-1)$	By subtraction

$$Q_i = y_{i...} + r\overline{y}_{....} - \frac{1}{p}R_i - \frac{1}{q}C_i \tag{10.3}$$

where $y_{i...}$ is the total for the ith treatment, $\overline{y}_{....}$ is the grand mean for the experiment, R_i is the sum of row totals that includes the ith treatment, and C_i is the sum of column totals that include the ith treatment.

The least squares estimate of the treatment effect is

$$\widehat{\tau}_i = \frac{pq(t-1)Q_i}{rt(p-1)(q-1)} \tag{10.4}$$

and the least squares estimate of the treatment mean is $\widehat{\mu}_i = \widehat{\mu} + \widehat{\tau}_i$, where $\widehat{\mu} = \overline{y}_{....}$ is the estimate of the general mean.

The efficiency factor for the nested row–column design is

$$E = \frac{t(p-1)(q-1)}{pq(t-1)} \tag{10.5}$$

where for the balanced lattice square the efficiency factor is $E = (k-1)/(k+1)$, with $p = q = k$ and $t = k^2$. For example, with $k = 3$ in Display 10.2, the efficiency factor is $E = 2/4 = .50$. The balanced lattice square would have to reduce experimental error variance by 50% to estimate comparisons between treatment means as precisely as a complete block design of the same size.

10.3 Cyclic Designs Simplify Design Construction

Extensive tables of design plans must be available to use the incomplete block designs discussed to this point. Setting up any design in the physical facility required for the experiment demands constant vigilance. Care must be taken to avoid mistakes when treatments are assigned to experimental units and when data are recorded. Complex incomplete block designs can add further complication to the process if there is a need for constant reference to the design layout.

Cyclic designs offer a degree of simplicity in the construction of the design and implementation in the experimental process. Cyclic designs are generated from an initial block of treatments so it is not necessary to table complete plans of the experimental layout. Once the initial block is known the experimental plan can be generated, and the treatments can be randomly assigned to the numerical labels in the design.

How to Construct a Cyclic Design

The *cyclic* designation refers to the method used to construct the designs as well as to the association scheme among the treatments. The treatments are assigned to the blocks through a cyclic substitution of treatment labels from an initial generating block.

The method requires that the treatments be labeled as $(0, 1, 2, \ldots, t-1)$. The cycle begins with an initial block of size k required by the experiment. Succeeding blocks are obtained by adding 1 to each treatment label in the previous block. If a treatment label exceeds $(t-1)$ it is reduced by modulo t. The cycle continues until it returns to the configuration of the initial block.

Consider the design for six treatments in blocks of three units generated by cyclic substitution in Display 10.3. The six treatments are labeled $(0, 1, 2, 3, 4, 5)$ for the design. A cyclic design is constructed with the initial block $(0, 1, 3)$. The second block is constructed by adding 1 to each treatment label in the first block which gives $(1, 2, 4)$ as the configuration for the second block. Add 1 to each treatment label in the second block to provide $(2, 3, 5)$ as the treatments for the third block.

Design 10.3 A Cyclic Design for Six Treatments in Blocks of Three Units

Block	
1	(0, 1, 3) initial block
2	(1, 2, 4)
3	(2, 3, 5)
4	(3, 4, 0)
5	(4, 5, 1)
6	(5, 0, 2)

The transition from block 3 to block 4 in the cycle requires one label in block 4 to be reduced by modulo $t = 6$. If 1 is added to each label in block 3 the configuration of block 4 will be $(3, 4, 6)$ where $5 + 1 = 6$ exceeds $(t-1) = 5$. Therefore, 6 must be reduced modulo 6, which is $6 = 0 (\text{mod } 6)$, and 0 replaces 6 in the block 4 configuration to give $(3, 4, 0)$. A similar replacement occurs in the transition from block 5 to block 6.

A full set of $b = 6$ blocks is constructed in the complete cycle shown in the display. The numbers make one complete cycle through the treatment labels down each of the columns in the display. The treatment labels in the first position of the blocks cycle from 0 to 5 while the second position cycle begins with the label 1 and the third position cycle begins with the label 3. The design is complete with blocks 1 through 6 since the seventh member of the cycle would be the initial block configuration, $(0, 1, 3)$.

Construct Row–Column Designs with the Cyclic Method

Cyclic designs can be used for row–column designs if the experiment requires two blocking criteria. The only requirement is that the number of replications be some multiple of the block size, or $r = ik$, where i is an integer value. The design in Display 10.3 with $t = 6$ and $r = k = 3$ generated from the initial block $(0, 1, 3)$ can be used as a row–column design with six rows and three columns.

The design is a column orthogonal design since each treatment appears in each of the columns. If the row and column integrity is maintained upon randomization, then a sum of squares for columns and rows is partitioned out of the total sum of squares. The treatments are adjusted for row blocks in the analysis.

Tables to Construct Cyclic Designs

A compact table of initial blocks required for relatively efficient cyclic designs with $4 \le t \le 15$ treatments, from John (1987), can be found in Appendix Table 10A.1. An additional table for $16 \le t \le 30$ treatments also was provided by John (1987). Other tables can be found in John (1966).

For a given block size k, the first $k-1$ treatments in the initial block are all taken to be the same for any number of treatments t. The final treatment in the block is taken as the corresponding value in the column for t in Appendix Table 10A.1. For example, with $k = 3$, $r = 3$, the initial block for $t = 5$ is $(0, 1, 2)$ and for $t = 6$ the initial block is $(0, 1, 3)$.

If a design with more replications is required $(r > k)$, a second- or third-generating block is given in the table. A set of blocks is constructed from each by cyclic substitution. The design with $t = 6$ treatments in blocks of size $k = 3$ generated with the initial block $(0, 1, 3)$ had $r = 3$ replications. The design with $r = 6$ replications is constructed by adding the initial block $(0, 2, 1)$ found in Table 10A.1 on the line with $k = 3$ and $r = 6$. The resulting design has $b = 12$ blocks, half of them generated from each of the initial blocks as shown in Display 10.4. The design is partially balanced with $\lambda_1 = 3$ and $\lambda_2 = 2$ for the two associate classes.

Display 10.4 A Cyclic Design Generated from Two Initial Blocks

Block		*Block*	
1	(0, 1, 3) initial block	7	(0, 2, 1) initial block
2	(1, 2, 4)	8	(1, 3, 2)
3	(2, 3, 5)	9	(2, 4, 3)
4	(3, 4, 0)	10	(3, 5, 4)
5	(4, 5, 1)	11	(4, 0, 5)
6	(5, 0, 2)	12	(5, 1, 0)

α Designs for Versatile Resolvable Designs from Cyclic Construction

The α designs are resolvable designs developed by Patterson and Williams (1976) in response to the requirements for statutory field trials of agricultural field crop varieties in the United Kingdom. The number of varieties and replications were fixed by statutory requirements and were not under the control of the experimenters. The large number of varieties necessitated incomplete block designs, and resolvable designs were required for proper management in the field. Existing resolvable designs did not always accommodate the number of varieties or block sizes required by the trials.

α Design Features

The designs have no limitation on block size except for the constraint that the number of treatments, t, must be a multiple of block size k, so that $t = sk$, to have a resolvable design with equal block sizes. Tabled plans of the complete designs are unnecessary since they can be generated by cyclic substitution from an initial array of numbers much like the cyclic designs.

The development of the α designs greatly increased the number and flexibility of resolvable designs available for experimental trials. The designs originally developed by Patterson and Williams (1976) for the statutory trials included designs with $r = 2$, 3, or 4 replications with $t \leq 100$ treatments and block sizes $4 \leq k \leq 16$.

The designs are partially balanced with two or three associate classes. The designs with two associate classes having $\lambda_1 = 0$ and $\lambda_2 = 1$ are denoted $\alpha(0, 1)$ designs, and designs with three associate classes having $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$ are denoted $\alpha(0, 1, 2)$ designs. Most of the designs considered for the variety trials were $\alpha(0, 1)$ designs. Because of the manner in which the designs are constructed neither balanced designs nor $\alpha(1, 2)$ designs exist.

The upper bound for the efficiency of the α designs provided by Patterson and Williams (1976) is

$$E = \frac{(t-1)(r-1)}{(t-1)(r-1) + r(s-1)} \tag{10.6}$$

How to Construct α Designs

The construction of a design for $t = sk$ treatments in r replications begins with a $k \times r$ array designated as the *generating* α array. Each column of the generating α array is utilized to construct $s - 1$ additional columns by cyclic substitution. The new $k \times rs$ array is denoted as the *intermediate* α^* array.

Consider a 4×3 generating α array for $t = 12$ treatments, $k = 4$ units per block, $r = 3$ replications, and $s = 3$ blocks per replication.

Generating α Arrays

Column		
1	*2*	*3*
0	0	0
0	0	2
0	2	1
0	1	1

Each column is used to generate $s - 1 = 2$ additional columns by cyclic substitution with mod(s) = mod(3) to derive the intermediate α^* array. The result will be $r = 3$ arrays of size $k \times s$. In this example, there will be three 4×3 arrays. The intermediate α^* arrays generated from each of the three columns of the α array are

Intermediate α^* Arrays

Array Generated From								
Column 1			*Column 2*			*Column 3*		
0	1	2	0	1	2	0	1	2
0	1	2	0	1	2	2	0	1
0	1	2	2	0	1	1	2	0
0	1	2	1	2	0	1	2	0

Finally, treatment labels are obtained by adding $s = 3$ to each element of the second row of the intermediate α^* array, $2s = 6$ to each element of the third row, and $3s = 9$ to the fourth row. The columns of the arrays are the blocks of the design, and each set of s columns generated from a generating α array column constitutes a complete replication. The resulting $\alpha(0, 1, 2)$ design is shown in Display 10.5.

The treatment labels for the design are $(0, 1, 2, \ldots, t - 1)$, and the actual treatments are randomly allocated to the labels. In addition, randomization proceeds with a random allocation of the design blocks to the actual blocks within the

Design 10.5 An α Design for 12 Treatments in Three Replication Groups

Replicate	I			II			III		
Block	*1*	*2*	*3*	*1*	*2*	*3*	*1*	*2*	*3*
	0	1	2	0	1	2	0	1	2
	3	4	5	3	4	5	5	3	4
	6	7	8	8	6	7	7	8	6
	9	10	11	10	11	9	10	11	9

replicate groups and a random allocation of the treatments to the actual experimental units within the physical blocks.

A table of 11 basic α arrays reproduced from Patterson, Williams, and Hunter (1978) is shown in Appendix Table 10A.2. A total of 147 α designs can be generated from the 11 arrays. The arrays can be used to generate designs for treatment numbers $20 \leq t \leq 100$ with $r = 2,\ 3$, or 4 replications and the usual constraint that $t = sk$. The arrays will produce designs with block sizes $k \geq 4$ with the condition $k \leq s$.

Consider the first array listed in Appendix Table 10A.2 for $s = k = 5$. The array has $k = 5$ rows and $r = 4$ columns.

0	0	0	0
0	1	4	2
0	2	3	4
0	3	2	1
0	4	1	3

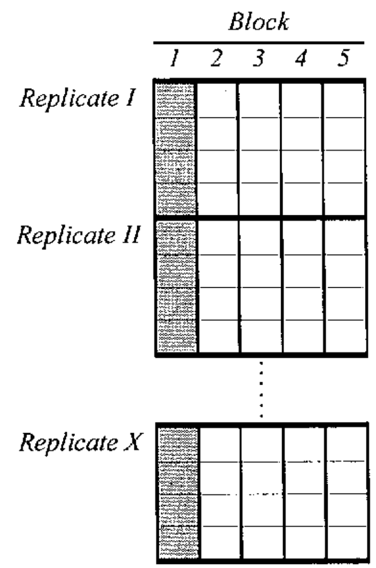
This particular array can be used to generate an α design with $r = 2$ replications by utilizing only the first two columns of the array. Likewise, designs for three or four replications can be generated by using the first three or all four columns, respectively. A design with block size $k = 5$ is obtained by utilizing all five rows of the array. A design with block size $k = 4$ is constructed by utilizing the first four rows of the array. With $s = 5$ a design may then be constructed for $t = 25$ treatments in blocks of $k = 5$ units with $r = 2,\ 3$, or 4 replications. Also with $s = 5$ a design may be constructed for $t = 20$ treatments in blocks of $k = 4$ units with $r = 2,\ 3$, or 4 replications.

Latinized Resolvable Incomplete Block Designs

The resolvable designs discussed to this point have a nested blocking structure. The incomplete blocks are randomized independently within each replicate whether the designs have a single blocking criterion or two blocking criteria with rows and columns.

In some instances, the blocks of one replicate may have a relationship to blocks in another replicate. These relationships could exist in experiments wherein the blocks in different replicates are contiguous in space, as illustrated in Display 10.6.

Display 10.6 Illustration of Contiguous Blocks in the Replicates



If the spatial layout is that shown in Display 10.6, then any single block—e.g., Block 1—in each replicate occurs in the same spatial line as Block 1 in each of the other replicates. The result is the occurrence of a "long" block in the vertical direction of the layout in Display 10.6.

If blocks are randomized separately in each replicate, it is entirely possible for all replications of one or more of the treatments to occur in the same "long" block. This could create an undesirable circumstance in the case of unforeseen disturbances which negated the use of the "long" block. To avoid this possibility Harshbarger and Davis (1952) introduced *Latinized* rectangular lattice designs. The rectangular lattice was structured such that each treatment occurred only once in each long block.

Williams (1986) presented results on more general designs in which treatments do not occur more than one time in the long blocks and extended the Latinized designs to include row–column blocking. The α design can be used to generate the more general Latinized designs as illustrated by John and Williams (1995) with a Latinized row–column design shown in Display 10.7. Notice that treatments appear no more than one time in each of the long blocks (columns). John and Williams (1995) provide a thorough discussion of cyclic and resolvable designs.

Display 10.7 A Latinized Row–Column Design

	Block			
	1	*2*	*3*	*4*
Replicate I	0	1	2	3
	8	5	10	7
	4	9	6	11
Replicate II	6	0	3	5
	1	7	4	2
	11	10	9	8
Replicate III	9	8	0	1
	7	6	5	4
	2	3	11	10

10.4 Choosing Incomplete Block Designs

A successful experiment will include the correct treatment design to address specific research questions and provide precise estimates of the treatment means and contrasts of interest to the investigator. The incomplete block design often is necessary to obtain the best possible conditions for comparisons among the treatments with as homogeneous groups of experimental units as possible.

The investigator's knowledge of the experimental conditions and material is indispensable in constructing suitable block sizes and composition. Each experiment presents its own challenges and conditions, and no set rules provide the correct design for every experiment. A knowledgeable investigator can craft a successful experiment with the intelligent application of sound design principles to the experimental material.

The guiding principle in selecting or constructing a design for a specific research problem is to make every attempt to choose a design that accommodates the research problem. In most cases, appropriate designs are available to accommodate the requirements of the experiments because such a large number of published plans for balanced incomplete block designs are available in the literature as a result of the research on design construction.

The systematic methods to construct balanced incomplete block designs require mathematics beyond the scope of this book. Discussions of research on design existence and construction can be found in John (1971), John (1987), and John and Williams (1995).

Where to Find Design Plans

Computer programs provide the most convenient method to develop an incomplete block design plan; CycDesigN (Whitaker, Williams, & John, 1998) and ALPHA+ (Williams & Talbot, 1993) are two examples of programs that generate designs discussed in Chapters 9 and 10.

Tabled plans for small experiments with $t \leq 11$ treatments are found in Appendix 9A.1. Plans for other numbers of treatments are given in Cochran and Cox (1957), Cox (1958), Fisher and Yates (1963), Box, Hunter, and Hunter (1978), and Mason, Gunst, and Hess (1989). Plans for balanced and partially balanced lattice, rectangular lattice, and lattice square designs may be found in Cochran and Cox (1957) and Petersen (1985). Plans for designs with row–column blocking as row orthogonal incomplete Latin squares may be found in Youden (1940), Cochran and Cox (1957), Cox (1958), and Petersen (1985). Some of the plans for row orthogonal designs are given in Appendix 9A.2. A large number of partially balanced incomplete block designs can be found in Clatworthy (1973) and Bose, Clatworthy, and Shrikande (1954).

An Informal Approach to Design Construction

In the cases where a standard design does not accommodate the research problem, it may be necessary to find an innovative design solution to avoid alterations in the treatment design. The need for occasional innovation argues for a more informal approach to experiment design (Mead, 1988). The primary reason for advocating the informal design is to avoid the trap of changing the treatment design or the research problem to fit some particular tabled design. Changing a research problem to fit the experiment design would be unwise under any circumstances. However, it is advantageous to understand some of the more formal properties of experiment design to fully appreciate innovative approaches and avoid possible pitfalls of the informal approach. The investigator must be quite familiar with the experimental conditions and material to construct a design appropriate to the study.

The construction can be quite informal if great care is taken in the placement of treatments with experimental units to avoid mishaps. Particular attention must be paid to the amount of information available for various treatment comparisons as the design is constructed. The relative differences in variances of the pertinent treatment comparisons must be evaluated prior to actual experimentation. Such an approach is advocated by Mead (1988).

The Efficiency Factor Increases with Block Size

We expect the experimental error variance to decrease as the block size decreases. The decrease in σ^2 with smaller block sizes depends on the success we have in placing the experimental units in the correct blocking arrangement. However, a close inspection of the efficiency factor suggests taking as large a block size as possible. The efficiency measure for a balanced incomplete block design can be expressed as

$$E = \frac{\lambda t}{rk} = \frac{t}{t-1}\left(1 - \frac{1}{k}\right) \tag{10.7}$$

The value of E increases with increasing block size k. A cursory scan of the list of balanced incomplete block designs in Appendix 9A.1 will confirm this relationship. A similar relationship is true for the partially balanced incomplete block designs.

Let Resources Dictate Block Size

The available resources dictate the choice of block size. In some experiments the block size is constrained to the available instruments for the experiment as was the case for the methyl glucoside vinylation experiment in Example 9.2. The investigator only had three chambers available to test five pressure treatments. With $t = 5, r = 6, k = 3$, and $\lambda = 3$, the efficiency factor for the balanced incomplete block design was $E = 3(5)/6(3) = 0.83$. Equivalent precision to a complete block design with five pressure chambers would require a 17% reduction in σ^2 for the incomplete block design. However, the only choice available to the investigator was the incomplete block design with three treatments per block.

Some experimental conditions allow greater flexibility in the choice of block sizes. Variety yield trials are placed in field plots on experimental farmland. In many of these situations the block sizes can be of any convenient size. The guiding principle is to have a sensible block size that provides as much homogeneity among the plots within blocks as is necessary for precise comparisons among the varieties. In this case the block size should be as large as possible while maintaining the required precision for variety comparisons.

Partially Balanced Designs Increase Efficient Use of Resources

Partially balanced designs can be used in place of balanced designs to utilize experimental resources more efficiently. Consider the partially balanced design in Display 9.3. The block sizes in the experiment were constrained to four experimental units. A balanced design would have required ten replications of the six treatments and 15 blocks. The 3 blocks of the partially balanced design provide two replications. If more replications are required the partially balanced design can be repeated. Unless ten replications are required the partially balanced design requires fewer resources than the balanced design.

The investigator had to make some sacrifice to have the smaller experiment. The partially balanced design provides more precision for some comparisons than others because not all the treatment pairs occur in the same number of blocks. However, with a prudent choice of design it is still possible to have a partially balanced design that provides near balance in all comparisons.

The partially balanced design in Display 9.3 had some treatment pairs occurring in $\lambda_1 = 2$ blocks and other treatment pairs occurring in $\lambda_2 = 1$ block. The efficiency for the comparisons among first and second associates can be shown to be $E_1 = 1.00$ and $E_2 = 0.86$, respectively, with an average efficiency of $E = 0.88$ (Clatworthy, 1973). There is near equal precision for both sets of associates, and only a small amount of precision is sacrificed with the partially balanced design.

α Designs Relieved Constraints for Large Experiments

The incomplete block designs originated with experimental trials that included relatively large numbers of treatments such as variety trials. The primary objectives of the incomplete block designs were to reduce the block sizes to reduce experimental error variance and to have experiments that could be managed on a replicate group basis. In agricultural field trials the block sizes were somewhat flexible as long as a reduction in block size managed to reduce the error variance. The original square and rectangular lattice designs provided a degree of relief for these situations. The variety numbers for these trials were still quite restrictive even though it often was possible for the investigators to adjust the number of included varieties without dire consequences.

The α designs relieved the constraint on the number of treatments for resolvable incomplete block designs to a considerable extent. They do still restrict the number of treatments to be a product of the number of units per block and the number of blocks in a replication group. However, it is now possible to have a resolvable design for a greater variety of treatment, block, and replication sizes. In addition, the α designs can be generated from a relatively simple set of tables.

Factorial Treatment Designs Require Special Attention

Structured treatment designs such as factorials require special attention when constructing incomplete block designs. The construction of incomplete block designs with factorial effects confounded with blocks will be considered in Chapter 11. These designs are particularly useful for the 2^n series of factorials that have many factors because no great sacrifice results from completely confounding large order interactions with blocks.

Cyclic Designs Are Helpful with Unstructured Treatment Designs

Balanced designs or resolvable designs can in some circumstances strain available resources because of the required number of replications. The more flexible partially balanced designs can be used in place of the balanced design with little loss in precision. However, the tables of partially balanced designs may not be readily available because many of them appear in older publications. Many of the tabled designs found in these publications can be constructed by cyclic substitution. Cyclic designs may be one of the best alternatives for producing an efficient design. They provide designs of many different sizes and are constructed easily from one or two initial blocks of treatments. The tables of initial blocks for cyclic designs appear in somewhat more recent publications than the other classes of designs.

EXERCISES FOR CHAPTER 10

1. A horticulturalist conducted a field test of $t = 8$ broccoli varieties with a resolvable balanced incomplete block design in a field experiment. The design had $k = 4$ plots per block and $s = 2$ blocks per replication. There were $r = 7$ replicate groups in the design. The data are pounds of broccoli harvested per plot. (R = replication, B = block within replication, T = variety)

R	*B*	*T*	*Yield*	*R*	*B*	*T*	*Yield*	*R*	*B*	*T*	*Yield*
1	1	1	46.5	3	2	2	52.7	6	1	1	45.6
1	1	2	55.7	3	2	4	52.5	6	1	3	37.0
1	1	3	37.7	3	2	5	46.4	6	1	5	49.5
1	1	4	50.3	3	2	7	46.3	6	1	7	45.4
1	2	5	43.1	4	1	1	40.7	6	2	2	48.2
1	2	6	47.6	4	1	4	53.0	6	2	8	54.0
1	2	7	35.5	4	1	6	45.0	6	2	4	47.4
1	2	8	45.9	4	1	7	38.0	6	2	6	53.8
2	1	1	40.1	4	2	2	56.1	7	1	1	44.6
2	1	2	55.8	4	2	3	39.0	7	1	4	52.4
2	1	7	39.7	4	2	5	54.7	7	1	5	50.2
2	1	8	51.7	4	2	8	48.5	7	1	8	52.0
2	2	3	41.2	5	1	1	44.1	7	2	2	56.8
2	2	4	61.7	5	1	2	56.6	7	2	3	37.8
2	2	5	49.8	5	1	5	44.7	7	2	6	45.7
2	2	6	53.6	5	1	6	51.7	7	2	7	42.6
3	1	1	42.3	5	2	3	39.0				
3	1	3	43.8	5	2	4	47.8				
3	1	6	45.6	5	2	7	41.6				
3	1	8	51.0	5	2	8	49.4				

a. Write a linear model for this experiment, describe the terms, and conduct the intrablock analysis.
b. Compute the standard error estimate of the difference between least squares estimates of two treatment means.
c. What is the efficiency factor for this design?
d. Write a summary of your analysis results and evaluation of the design.

2. An agronomist conducted an alfalfa variety trial in a quadruple lattice design. There were 25 varieties grown in four replicate groups. Each replicate group consisted of five blocks of five varieties in a square lattice. The yield data that follow are pounds of alfalfa hay harvested per plot. The variety numbers appear in parentheses.

Block	*Replicate I*				
1	19.1 (2)	20.4 (1)	23.2 (4)	19.3 (3)	21.4 (5)
2	18.6 (8)	18.3 (6)	21.3 (10)	12.0 (9)	19.3 (7)
3	20.8 (15)	19.5 (11)	20.8 (13)	20.3 (12)	19.0 (14)
4	21.0 (17)	19.4 (16)	19.7 (19)	17.5 (18)	20.2 (20)
5	19.6 (24)	19.0 (23)	19.4 (21)	20.6 (25)	20.3 (22)

Block	*Replicate II*				
1	19.4 (11)	19.8 (21)	18.4 (6)	21.5 (1)	19.7 (16)
2	18.6 (17)	20.8 (2)	20.9 (12)	19.7 (7)	20.0 (22)
3	16.6 (18)	19.8 (3)	18.2 (23)	19.1 (8)	20.7 (13)
4	21.8 (4)	20.6 (14)	20.0 (19)	16.8 (9)	19.5 (24)
5	21.3 (15)	19.6 (20)	20.1 (25)	20.4 (10)	20.7 (5)

Block	*Replicate III*				
1	21.1 (1)	20.1 (13)	20.3 (7)	20.8 (25)	22.2 (19)
2	19.5 (21)	19.1 (14)	20.5 (20)	20.5 (2)	20.3 (8)
3	13.5 (9)	23.1 (22)	20.5 (3)	21.2 (15)	18.6 (16)
4	22.6 (10)	17.2 (23)	22.2 (4)	17.0 (17)	17.8 (11)
5	20.0 (24)	17.7 (18)	18.7 (5)	19.8 (12)	16.4 (6)

Block	*Replicate IV*				
1	20.2 (12)	22.6 (1)	19.5 (23)	22.4 (20)	13.6 (9)
2	22.5 (24)	21.9 (16)	22.5 (10)	21.1 (13)	19.3 (2)
3	19.2 (25)	20.0 (3)	17.1 (6)	21.0 (17)	18.9 (14)
4	21.7 (15)	20.0 (21)	18.2 (7)	17.3 (18)	20.9 (4)
5	17.3 (8)	19.3 (22)	20.7 (5)	19.2 (11)	16.9 (19)

Source: Dr. M. Ottman, Department of Plant Sciences, University of Arizona.

a. Write a linear model for the experiment, explain the terms, and conduct the intrablock analysis for a simple, triple, or quadruple lattice design (instructor discretion).
b. Compute the standard error estimates of the difference between intrablock estimates of two variety means that are first associates and between those that are second associates. Also compute an average standard error.
c. What is the efficiency factor for the simple or triple lattice design? Interpret the efficiency factor.
d. Write a summary of your analysis results and evaluation of the design.

3. An agronomist conducted a wheat variety test in a balanced lattice square design. There were $t = 9$ varieties in a 3×3 balanced lattice square in $r = 4$ replicate groups. The wheat yields are shown in the table that follows in the row–column arrangements for each replicate group. Variety numbers appear in parentheses.

Replicate I

53.5 (6)	53.2 (4)	57.7 (5)
53.1 (3)	58.6 (1)	53.9 (2)
57.2 (9)	55.0 (7)	51.5 (8)

Replicate II

53.7 (4)	53.6 (2)	57.8 (9)
54.5 (3)	52.8 (7)	53.3 (5)
48.9 (8)	53.5 (6)	56.7 (1)

Replicate III

49.4 (8)	54.7 (4)	55.6 (3)
54.5 (6)	54.2 (2)	54.4 (7)
59.7 (1)	55.7 (9)	54.1 (5)

Replicate IV

54.0 (7)	57.2 (1)	53.2 (4)
56.9 (9)	54.8 (3)	55.4 (6)
48.9 (8)	53.4 (2)	55.9 (5)

a. Write a linear model for this experiment, describe the terms, and conduct the intrablock analysis for this experiment.
b. Compute the standard error estimate of the difference between least squares estimates of two treatment means.
c. What is the efficiency factor for this design?
d. Write a summary of your analysis results and evaluation of the design.

4. A variety trial was conducted in an $\alpha(0,1,2)$ resolvable design. There were $t = 18$ varieties in $r = 4$ replicate groups. There were $s = 3$ blocks of $k = 6$ varieties in each replicate. The yield data follow with the variety numbers in parentheses. Varieties 1 and 5 were control varieties.

Block	*Replicate I*					
1	88.2 (5)	82.5 (10)	84.3 (15)	87.0 (6)	84.5 (12)	88.9 (8)
2	82.4 (1)	82.9 (14)	83.1 (3)	84.7 (13)	83.3 (16)	89.0 (4)
3	93.1 (2)	82.7 (11)	88.9 (17)	88.6 (18)	84.1 (9)	87.5 (7)

Block	*Replicate II*					
1	85.4 (4)	73.0 (11)	84.2 (7)	80.3 (14)	79.6 (10)	86.0 (6)
2	87.9 (8)	85.1 (9)	79.4 (18)	80.7 (13)	89.3 (5)	81.5 (3)
3	82.4 (1)	88.5 (2)	87.0 (12)	85.4 (17)	85.9 (15)	79.1 (16)

Block	*Replicate III*					
1	83.6 (6)	79.4 (17)	81.3 (4)	80.5 (9)	80.9 (8)	79.3 (1)
2	80.4 (7)	88.2 (5)	82.3 (14)	88.0 (12)	90.0 (2)	83.6 (3)
3	81.4 (18)	84.8 (15)	81.0 (10)	81.2 (13)	79.1 (11)	83.8 (16)

Block	*Replicate IV*					
1	80.5 (16)	77.1 (11)	84.4 (17)	90.4 (6)	82.9 (14)	83.0 (12)
2	87.9 (8)	78.9 (18)	81.4 (1)	83.5 (2)	82.2 (15)	79.0 (3)
3	84.2 (7)	83.0 (10)	87.6 (9)	81.7 (13)	91.3 (5)	87.4 (4)

Source: P. Seeger, Department of Statistics, The Swedish University of Agricultural Sciences.

a. Identify the first, second, and third associates of variety 1, where $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 2$. How many of each are there, and which varieties are in each group of associates?

b. Write a linear model for the experiment, explain the terms, and conduct the intrablock analysis to obtain intrablock estimates of the variety means and their standard error estimates.
c. The α designs have a multiplicity of standard errors for the differences between pairs of estimated treatment means. If your program is capable, compute the standard error estimates of the difference between intrablock estimates of the control variety means, varieties 1 and 5, and the other varieties; the average standard error for the first, second, and third associates; and an overall average standard error.
d. Compute the standard error estimates of the difference between the variety means; an average standard error of the difference for first, second, and third associates; and an overall average standard error.
e. Write a summary of your analysis results and evaluation of the design.

5. A human factors study is to be conducted on speed of perception relative to object shape in statistical graphs. Eight shapes will be used as treatments in the study. It is thought that a subject should evaluate no more than four shapes in any one sitting. Researchers have decided to use an incomplete block design with subjects as blocks.
a. Construct a cyclic design with $r = 4$ replications of $t = 8$ object shape treatments. Use human subjects as blocks with $k = 4$ treatments per subject.
b. Randomize the blocks of treatment labels to subjects, and randomize the actual object shape treatments to the treatment labels.
c. Suppose the order of presentation to the subjects could be an important source of variation. Construct the design as a row–column design, and randomize accordingly.
d. Suppose eight replications were required for the study; construct the cyclic design with eight replications.

6. Generate an α design for three replications of 35 treatments in blocks of five plots.
a. Show the generating α array, the intermediate α arrays, and the final alpha design with treatment labels.
b. Randomize the incomplete blocks of treatment labels to actual blocks, and randomize the treatment labels to the units within a block.
c. Randomly assign the actual treatments to the treatment labels.

10A.1 Appendix: Plans for Cyclic Designs

Table 10A.1 Initial blocks to generate efficient cyclic partially balanced incomplete block designs for $4 \leq t \leq 15, r \leq 10$

			kth treatment, t =											
k	*r*	*First k − 1 treatments*	4	5	6	7	8	9	10	11	12	13	14	15
	2	0	1	1	1	1	1	1	1	1	1	1	1	1
	4	0	2	2	2	3	3	3	3	3	3	5	4	4
2	6	0	1	3	3	2	2	2	2	5	5	2	6	2
	8	0	1	4	5	4	4	4	4	2	2	4	3	7
	10	0	2	1	4	5	5	5	5	4	4	3	5	5
	3	0 1	2	2	3	3	3	3	4	4	4	4	4	4
3	6	0 2	1	3	1	3	7	6	7	7	5	7	7	8
	9	0 1	3	2	3	3	4	5	3	3	6	4	6	5
4	4	0 1 3	–	2	2	6	7	7	6	7	7	9	7	7
	8	0 1 4	–	2	2	6	7	8	2	6	6	6	6	6
	5	0 1 2 4	–	–	5	5	7	7	7	7	7	7	9	10
5	10	{ 0 2 3 4	–	–	5	5	7	8	9	8				
		{ 0 2 3 6									7	11	12	10
6	6	0 1 2 3 6	–	–	–	5	5	5	5	10	10	10	10	10
7	7	0 1 2 3 4 7	–	–	–	–	5	5	9	9	9	9	9	10
8	8	0 1 2 3 4 6 8	–	–	–	–	–	5	9	9	9	9	11	11
9	9	0 1 2 3 4 5 7 9	–	–	–	–	–	–	8	8	8	10	10	10
10	10	0 1 2 3 4 5 6 9 10	–	–	–	–	–	–	–	7	7	7	12	12

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10A.2 Appendix: Generating Arrays for α Designs

Table 10A.2 Generating Arrays ($k \times r$) for α Designs

$s = k = 5$

0	0	0	0
0	1	4	2
0	2	3	4
0	3	2	1
0	4	1	3

$s = k = 6$

0	0	0	0
0	1	5	4
0	3	2	5
0	2	3	1
0	4	1	2
0	5	1	3

$s = k = 7$

0	0	0	0
0	1	3	2
0	2	6	4
0	4	5	1
0	3	2	6
0	5	1	3
0	6	4	5

$s = k = 8$

0	0	0	0
0	1	2	6
0	3	7	1
0	5	3	4
0	2	5	3
0	4	1	6
0	6	0	2
0	7	6	5

$s = k = 9$

0	0	0	0
0	1	8	7
0	3	6	4
0	7	2	3
0	2	3	5
0	4	1	6
0	5	7	2
0	6	5	1
0	8	4	7

$s = k = 10$

0	0	0	0
0	1	9	5
0	3	6	9
0	5	7	2
0	4	5	6
0	6	3	1
0	7	2	4
0	8	4	7
0	9	8	2
0	2	6	3

$s = 11, k = 9$

0	0	0	0
0	1	6	7
0	4	8	1
0	9	7	5
0	2	3	6
0	5	1	3
0	6	5	10
0	3	9	4
0	7	4	1

$s = 12, k = 8$

0	0	0	0
0	1	2	3
0	7	5	1
0	9	6	4
0	4	11	8
0	11	3	10
0	10	4	7
0	5	1	6

$s = 13, k = 7$

0	0	0	0
0	1	4	10
0	3	8	11
0	9	2	1
0	12	10	6
0	8	5	12
0	6	7	8

$s = 14, k = 7$

0	0	0	0
0	1	8	10
0	9	10	7
0	11	13	2
0	2	6	1
0	5	11	12
0	3	1	11

$s = 15, k = 6$

0	0	0	0
0	1	8	7
0	3	12	14
0	7	2	5
0	10	13	11
0	14	3	8

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